THE

DOCTRINE

Geo. OF Baron

PROPORTION,

7/180.

ARITHMETICAL

Monkwearm.th AND 1790

GEOMETRICAL.

Together with a general Method of arguing by proportional Quantities.

Si Proportionis Doctrinam e Mathesi abstuleris, nihil fere præclarum aut egregium relinques.

Wh. Tac. Eucl.

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THE

PREFACE.

SINCE all manner of quantities require to be compared together, in mathematical computations, and their various relations ſearched out and determined; and as moſt of our knowledge in mathematical ſubjects depends on the proportions of ſeveral things to one another: ſo it is requiſite that the nature of proportion, and the methods of reaſoning thereby, be diſtinctly laid down and well underſtood. It is a method of reaſoning ſo very ſhort, ſubtle, ſolid, and certain, and likewiſe ſo uſeful in all parts of the mathematics, that it is impoſſible to make the leaſt progreſs without it. It is the marrow of the mathematics, and the very ſoul of geometry and geometrical reaſoning. Therefore it is abſolutely neceſſary, that every one who expects to ſucceed in his mathematical ſtudies, ſhould make himſelf acquainted with the nature of reaſoning with proportional quantities, and become ready and quick in the uſe thereof.

I had before, in the Treatiſe of Arithmetic, demonſtrated ſome few things relating to proportions; but no more than I had then preſent occaſion for, in treating of the properties of numbers. But in this ſmall tract, I have demonſtrated the doctrine of proportions univerſally, for all quantities whatſoever, as well as numbers.

The method I have here followed is this: Sect. I. treats of arithmetical proportion and progression. And Sect. II. of geometrical proportion. And herein I have taken the liberty to deviate from Euclid, *by giving a different definition of proportional quantities; his being abstruse and unintelligible, especially to young students. This here laid down being evidently agreeable to, and deducible from, the first, simple, and natural idea we form of proportion. Neither have I followed his order of propositions, or method of demonstration: but have omitted many of his propositions as of little use, and added several other more useful ones, which he had not. And these I have demonstrated from that most simple idea of proportion before mentioned, with the greatest ease and perspicuity imaginable. And because the method of arguing by a general proportion is vastly shorter and easier than the common way with four terms; therefore I have in Sect. III. demonstrated the fundamental propositions it depends on; and has shewn and explained the way of proceeding, according to that method. And therefore I hope this will both instruct and delight the diligent reader.*

W. Emerson.

A X I O M S.

1. The whole is equal to all the parts taken together.

2. If equal quantities be added to equal quantities; the ſums will be equal.

3. If equal quantities be taken from equal quantities; the remainders will be equal.

4. If equal quantities be equally multiplied; the products will be equal.

5. If equal quantities be divided by equal numbers; the quotients will be equal.

6. Equal quantities have the ſame proportion to any third quantity: and any quantity has the ſame ratio to equal quantities.

7. Thoſe quantities are equal, that have the ſame ratio to any third; or when a third has the ſame ratio to each of them.

8. Thoſe ratios or quantities, that are equal to a third, are equal to one another.

9. A greater quantity has a greater ratio to a third, than a leſſer quantity has. And that which has the greater ratio, is the greater quantity.

10. If there be two equal ratios, and one be greater than a third, the other will be greater; if leſs, the other will be leſs.

The signification of the Signs or Characters here used.

$+$ to be added.

$-$ to be subtracted.

$\times$ multiplied by, or AB is A multiplied by B.

$\div$ divided by, or $\frac{A}{B}$ is A divided by B.

$=$ equal to.

$::$ geometrical proportion, as $A : B :: C : D$, signifies A is to B, as C is to D.

$\propto$ is as; a mark of general proportion.

$\div\!\!\div$ continual proportion, or geometrical progression. As $A : B : C : D \div\!\!\div$, signifies that A is to B, as B to C, as C to D, &c.

$\cdot\!\cdot\!\cdot$ arithmetical proportion, as $A . B \cdot\!\cdot\!\cdot C . D$.

$\div$ arithmetical progression.

$.\!\cdot\!.\!\cdot\!.$ harmonic proportion.

$\div$ harmonic progression.

SECT. I.

Arithmetical Proportion.

DEFINITIONS.

1. *ARITHMETIC proportion*, is the relation that two quantities, of the fame kind, have to one another, in refpect of their difference. The former quantity is called the *antecedent*; and the latter, the *confequent*. And thefe are called the *terms* of the proportion.

2. *Ratio* is the difference between the antecedent and confequent. Therefore arithmetic ratio is of the fame kind as the quantities themfelves. This is commonly called the *common difference*.

3. *Quantities arithmetically proportional*, are thofe that have the fame arithmetic ratio, when compared two and two; fo that the antecedents, may be every where fubtracted from the confequents; or elfe the confequents from the antecedents.

4. Continued proportion is when the firft has the fame proportion to the fecond, as the fecond to the third.

5. *Arithmetical progreffion*, is when a feries of quantities are in the fame arithmetical proportion. Or when they increafe, or decreafe by equal differences.

6. *Mufical proportion, and progreffion*, is when there is a feries of quantities, where the numerators are the fame, and the denominators in arithmetic progreffion.

PROP. I.

If four quantities are arithmetically proportional, A . B ∵ C . D; *the sum of the extremes is equal to the sum of the means,* $A + D = B + C$.

For $A - B = C - D$ (Def. 3), and adding $B + D$, $A - B + B + D = C + B - D + D$ (Ax. 2); that is, $A + D = C + B$.

Cor. *If three quantities be in arithmetic progression, the sum of the extremes is double the mean.*

PROP. II.

If there be two ranks of quantities in arithmetic proportion; their sums and differences shall also be in arithmetic proportion. If A . B ∵ C . D, *and* P . Q ∵ R . S; *then* $A + P \,.\, B + Q$ ∵ $C + R \,.\, D + S$, *and* $A - P \,.\, B - Q$ ∵ $C - R \,.\, D - S$.

For let $A - B = C - D = m$, and $Q - P = S - R = n$. Then $B = A - m$, $D = C - m$, $Q = P + n$, $S = R + n$. And $B + Q = A - m + P + n$, $D + S = C - m + R + n$. But $A + P \,.\, A - m + P + n$ ∵ $C + R \,.\, C - m + R + n$ (Def. 3).

Again, $B - Q = A - m - P - n$, and $D - S = C - m - R - n$. But $A - P \,.\, A - m - P - n$ ∵ $C - R \,.\, C - m - R - n$ (Def. 3).

PROP. III.

If three quantities are in arithmetic progression; the rectangle of the extremes, together with the square of the common difference, is equal to the square of the middle term. If A . B . C ∸, *then* $AC + \overline{B - A}^2 = BB$.

For

For let $D = B - A = C - B$, and $A = B - D$, $C = B + D$; then $AC = \overline{B-D} \times \overline{B+D} = BB + BD - BD - DD = BB - DD$. And $AC + DD = BB$ (Ax. 2).

Cor. *A set of arithmetical proportionals, whose common difference is exceeding small, is nearly a set of geometrical proportionals.* See the next section.

PROP. IV.

In a series of quantities in arithmetical progression; the sum of the extremes is equal to the sum of any two means, equally distant from the extremes. If $A . B . C . D . E . F . G \div$; *then* $A + G = B + F = C + E$, *&c.*

For since $A . B \because F . G$ (Def. 5), therefore $A + G = B + F$ (Prop. I). And since $B . C \because E . F$, therefore $C + E = B + F = A + G$, *&c.*

Cor. *Hence the sum of the extremes is double the mean, when the number of terms is odd.*

PROP. V.

If out of a series of quantities in arithmetical progression, there be taken any series of equidistant terms; this series will also be in arithmetic progression.

If $A . B . C . D . E . F . G . H . I . K . L . M \div$, *then* $B \quad E \quad H \quad L$ *are* $\div$.

For $C - B = D - C = E - D = R$, and adding all together, $E - B = 3R$.
Also $F - E = G - F = H - G = R$, and $H - E = 3R$.
Again, $I - H = K - I = L - K = R$, and $L - H = 3R$, *&c.*
Therefore, $E - B = H - E = L - H$ (Ax. 8).
And $B . E . H . L \div$ (Def. 5).

PROP.

PROP. VI.

In a ſeries of quantities in arithmetic progreſſion, A . B . C . D . E, *whoſe number is* n, *and common difference* x; *the laſt term* (E) $= A + \overline{n-1} \times x$ *in an increaſing progreſſion, or laſt term* (E) $= A - \overline{n-1} \times x$ *in a decreaſing one.*

For the difference between A and B, B and C, C and D, D and E, being x; the difference between A and E will be ſo many times x, as are the terms beyond A; that is, $\overline{n-1} \times x$. Whence $\overline{A-E}$, or $E - A = \overline{n-1} \times x$. And $E = A + \overline{n-1} \times x$, or $= A - \overline{n-1} \times x$ (Ax. 2, 3).

Cor. *The common difference is equal to the difference of the extremes, divided by the number of terms leſs one.*

For $x = \dfrac{A - E \text{ or } E - A}{n-1}$ (Ax. 5).

PROP. VII.

The ſum, of a ſeries of quantities in arithmetic progreſſion, is equal to half the product, of the ſum of the extremes, multiplied by the number of terms.

If A . B . C . D . E $\div$, *then the ſum* $= \dfrac{\overline{A+E} \times n}{2}$, *n being the number of terms.*

For	A	+ B	+ C	+ D	+ E	= ſum
And	E	+ D	+ C	+ B	+ A	= ſum

Adding, $A + E + B + D + C + C + B + D + A + E =$ twice the ſum.
That is, $A + E + A + E + A + E + A + E + A + E$ (Prop. IV).

Therefore twice the ſum is equal to as many times $A + E$, as there are terms, or the ſum $= \dfrac{A + E}{2} \times n$.

PROP.

PROP. VIII.

In a series of quantities in arithmetical progression from o, their differences are equal; in their squares, the differences of the differences, or the second differences, are equal; in their cubes, the third differences are equal; and so on.

Let the series be o, a, $2a$, $3a$, $4a$, $5a$, $6a$, *&c.*

then	o,	aa	$4aa$	$9aa$	$16aa$	$25aa$ *&c.*	squares.
		aa	$3aa$	$5aa$	$7aa$	$9aa$	1 differences.
			$2aa$	$2aa$	$2aa$	$2aa$	2 differences.

Again,	o	a^3	$8a^3$	$27a^3$	$64a^3$	cubes.
		a^3	$7a^3$	$19a^3$	$37a^3$	1 differences.
			$6a^3$	$12a^3$	$18a^3$	2 differences.
				$6a^3$	$6a^3$ &c.	3 differences.

And so for higher powers.

Cor. 1. *In the* n^{th} *powers, the* $\overline{n+1}^{th}$ *differences are* o.

Cor. 2. *The equal differences in the laterals, squares, cubes, biquadrates,* &c. *are* $1a$, $1\times 2aa$, $1\times 2\times 3a^3$, $1\times 2\times 3\times 4a^4$, &c. *respectively.*

SECT. II.

Geometrical Proportion.

DEFINITIONS.

1. GEOMETRICAL *proportion*, is the relation or reſpect, that two quantities, of the ſame kind, have to one another in regard to their bigneſs. The former quantity is called the *antecedent*; and the ſecond, the *conſequent*.

2. *Ratio* is the quotient ariſing by dividing the antecedent by the conſequent: Or it is the number which expreſſes how oft the antecedent contains the conſequent; which number may be either whole, fractional, or ſurd. When the antecedent and conſequent are equal; it is called a ratio of *equality*; if not, of *inequality*.

3. *Terms* of the ratio, are the antecedent and its conſequent.

4. *Proportional quantities* are thoſe that have the ſame ratio or proportion, when compared two and two together; that is, when the firſt is to the ſecond, as the third to the fourth; or when the firſt contains the ſecond, as oft as the third contains the fourth; and the contrary.

5. *Homologous or alternate terms*, are the antecedents of ſeveral ratios, or elſe the conſequents. And any antecedent and its conſequent, are called *analogous terms*.

6. *Direct proportion*, is when the ſame proportion holds from the firſt term to the ſecond, and from

from the third to the fourth, as if A, B, C, D, be four quantities; then it is *directly* A : B :: C : D.

7. *Reciprocal or inverſe proportion*, is when one ſort of quantity increaſes, in the ſame proportion that another decreaſes.

8. *Diſcreet proportion*, is when out of four terms, the ſecond has not the ſame proportion to the third, which the firſt has to the ſecond, or the third to the fourth.

9. *Continual proportion*, is when the firſt term has the ſame proportion to the ſecond, as the ſecond to the third.

10. *Geometrical progreſſion*, is when a ſet of quantities are in continual proportion; or when the firſt has the ſame ratio to the ſecond, as the ſecond to the third, and as the third to the fourth, and the fourth to the fifth, *&c.*

11. *Extreme and mean ratio*, is when a quantity is ſo divided, that the leſſer part, the greater part, and the whole, are in continual proportion.

12. *Complicate ratio*, is that which ariſes by multiplying ſeveral other ratios together.

13. *Duplicate, triplicate, ratio*, &c. is the ſquare, cube, *&c.* of ſome given ratio.

14. *Harmonical ratio*, is when a quantity is divided into three parts, ſo that the whole is to one part, as the ſecond part to the third. And when the ſecond and third are equal; it is called *harmonic proportion continued.*

PROP.

PROP. I.

If several pairs of quantities are in the same proportion, A : B :: C : D :: E : F :: G : H; *then as any antecedent to its consequent, so is any other antecedent to its consequent,* A : B :: G : H.

For since $\frac{A}{B} = \frac{C}{D} = \frac{E}{F} = \frac{G}{H}$ (Def. 4), therefore $\frac{A}{B} = \frac{G}{H}$ (Ax. 8); whence A : B :: G : H (Def. 4).

PROP. II.

If four quantities are proportional, A : B :: C : D; *and if the first* A, *be greater than the second* B; *then the third* C, *shall be greater than the fourth* D. *If equal, they shall be equal; if less, less.*

For since $\frac{A}{B} = \frac{C}{D}$ (Def. 4), by the nature of fractional quantities, if A be greater than B, the quotient or ratio will be more than 1, and therefore C greater than D. But if A be equal to B, $\frac{A}{B} = 1$, and C = D. But if A be less than B, the quotient is less than 1, and therefore C less than D.

PROP. III.

If four quantities are proportional, A : B :: C : D; *they shall also be proportional by reversion; that is, the second* B *is to the first* A; *as the fourth* D, *is to the third* C; *or* B : A :: D : C.

For let $\frac{A}{B} = \frac{C}{D} = r$ the ratio, then $A = Br$, and $C = Dr$ (Ax. 4); and $B = \frac{A}{r}$, and $D = \frac{C}{r}$ (Ax. 5); also $\frac{B}{A} = \frac{1}{r}$, and $\frac{D}{C} = \frac{1}{r}$ (ib.); whence $\frac{B}{A} = \frac{D}{C}$ (Ax. 8); therefore B : A :: D : C (Def. 4).

PROP. IV.

If four quantities of the same kind are proportional, A : B :: C : D; *they shall be proportional alternately or by permutation; that is, the first* A, *shall be to the third* C; *as the second* B, *is to the fourth* D.

For let $\frac{A}{B} = \frac{C}{D} = r$, then A = B$r$, and C = D$r$ (Ax. 4); then $\frac{A}{C} = \frac{Br}{Dr} = \frac{B}{D}$ (Ax. 5); therefore A : C :: B : D (Def. 4).

PROP. V.

Quantities are in the same ratio, as their equimultiples; A : B :: nA : nB.

For let $\frac{A}{B} = r$, then A = Br (Ax. 4); and nA = nBr (ib.); and $\frac{nA}{nB} = r$ (Ax. 5); therefore $\frac{A}{B} = \frac{nA}{nB}$ (Ax. 8); therefore A : B :: nA : nB.

Cor. 1. *Quantities are in the same ratio, as their like parts.*

For nA : nB :: $\frac{nA}{n} : \frac{nB}{n}$ :: A : B.

Cor. 2. *The like parts of two quantities, taken an equal number of times, are as the quantities themselves.*

PROP. VI.

If four quantities are proportional, A : B :: C : D; *and two homologous or analogous terms be both of them equally multiplied, or divided; the four terms will still be proportional.*

For C : D :: nC : nD (Pr. V) :: $\frac{C}{n} : \frac{D}{n}$ (Pr. V. Cor. 1); therefore A : B :: nC : nD :: $\frac{C}{n} : \frac{D}{n}$ (Prop. I).

Again,

Again, $A : C :: B : D$ (Prop. IV) $:: nB : nD :: \frac{B}{n} : \frac{D}{n}$ (Prop. VI). Therefore $A : nB :: C : nD$. And $A : \frac{B}{n} :: C : \frac{D}{n}$ (Prop. IV).

Cor. 1. *If two correſpondent terms be multiplied by one number, and the other two terms by another number; the reſulting terms will be proportional: If* $A : B :: C : D$, *then* $mA : mB :: nC : nD$; *or* $mA : nB :: mC : nD$.

Cor. 2. *And if two correſpondent terms be divided by one number, and the other two by another number; the quotients will be proportional.*

Cor. 3. *Hence, inſtead of any two correſpondent terms; two others, proportional to them, may be put in their room.*

PROP. VII.

If four quantities are proportional; and inſtead of two factors, in two analogous terms, if there be ſubſtituted two other quantities, in the ſame ratio; the four quantities will ſtill be proportional: If $A : B :: PQ : RS$; *and* $Q : S :: M : N$. *Then* $A : B :: PM : RN$.

For ſince $A : B :: PQ : RS$; by dividing the antecedents by P, and the conſequents by R, $\frac{A}{P} : \frac{B}{R} :: Q : S :: M : N$ (Prop. VI. Cor.); then multiplying the antecedents by P, and the conſequents by R, we have $A : B :: PM : RN$.

PROP.

PROP. VIII.

If the parts taken away from two whole quantities, be as the wholes; then the remainders, ſhall be as the wholes. If A : C : : A + B : C + D; *then* B : D : : A + B : C + D.

For A : A + B : : C : C + D (Prop. IV); and A + B : A : : C + D : C (Prop. III); and $\frac{A+B}{A} = \frac{C+D}{C}$ (Def. 4); that is, $1 + \frac{B}{A} = 1 + \frac{D}{C}$, and $\frac{B}{A} = \frac{D}{C}$ (Ax. 3); therefore B : A : : D : C, and B : D : : A : C (Prop. IV) : : A + B : C + D.

Cor. *The ſame things ſuppoſed, the remainders ſhall be as the parts taken away,* A : B : : C : D.

PROP. IX.

The ſum of the greateſt and leaſt, of four proportional quantities, is greater than the ſum of the other two.

Suppoſe A : B : : C : D, and let A be the greateſt term, then of conſequence D is the leaſt (Prop. II): then $\frac{A}{B} = \frac{C}{D} = r$. Now ſince A is greater than B, r is greater than 1, therefore put $r = 1 + s$. Whence $A = rB = B + sB$, and $C = rD = D + sD$ (Ax. 8). Then $A + D = B + sB + D$, and $B + C = B + D + sD$. But B is greater than D, and sB greater than sD; therefore $B + D + sB$ is greater than $B + D + sD$; or $A + D$ greater than $B + C$.

Cor. *The ſum of* A *and* D = *ſum of* B *and* C + $\overline{r - 1} \times \overline{B - D}$.

 For

For one of these sums exceeds the other by $s \times \overline{B - D}$.

PROP. X.

If several quantities are proportional; A : B : : C : D : : E : F : : G : H; *as one of the antecedents, to its consequent*; *so is the sum of all the antecedents, to the sum of all the consequents*; A : B : A + C + E + G : B + D + F + H.

For let $\frac{A}{B} = r$, or $A = Br$, $C = Dr$, $E = Fr$, $G = Hr$, and $A + C + E + G = Br + Dr + Fr + Hr = \overline{B + D + F + H} \times r$ (Ax. 8); therefore $\frac{A + C + E + G}{B + D + F + H} = \frac{\overline{B + D + F + H} \times r}{B + D + F + H} = r$; therefore $\frac{A}{B} = \frac{A + C + E + G}{B + D + F + H}$; therefore, &c.

PROP. XI.

If there be two ranks of proportional quantities, and the two means be the same in both; *the extremes will be reciprocally proportional. If* A : B : : C : D, *and* E : B : : C : F; *then* A : E : : F : D.

For let $\frac{A}{B} = \frac{C}{D} = r$, and since B : E : : F : C (Pr. III); therefore let $\frac{B}{E} = \frac{F}{C} = s$. Then $rs = \frac{A}{B} \times \frac{B}{E} = \frac{C}{D} \times \frac{F}{C}$ (Ax. 4); that is, $\frac{A}{E} = \frac{F}{D}$; or A : E : : F : D.

Cor. *In two ranks of proportional quantities, if the extremes be the same in both*; *the means will be reciprocally proportional.*

For

For if B : A : : D : C, and B : E : : F : C; then by reverſion A : B : : C : D, and E : B : : C : F. Whence A : E : : F : D (Prop. XI).

PROP. XII.

If four quantities are proportional; A : B : : C : D; *the product of the extremes is equal to the product of the means,* AD = BC.

For let $\frac{A}{B} = \frac{C}{D} = r$, then A = B$r$, and C = D$r$ (Ax. 4); whence AD = BrD, and BC = BrD (Ax. 4); therefore AD = BC (Ax. 8).

Cor. 1. *If two products are equal,* AD = BC; *the ſides or factors are reciprocally proportional,* A : B : : C : D.

For let A : B : : C : Q, then AQ = BC (Prop. XII) = AD (hyp.); therefore Q = D (Ax. 5); and A : B : : C : D (Ax. 7).

Cor. 2. *If three quantities are continually proportional; the rectangle of the extremes is equal to the ſquare of the mean. And the contrary.*

Cor. 3. *In four proportional quantities, if one extreme be multiplied by any number, and the other extreme, divided by it; the quantities will ſtill be proportional. The ſame holds of the means. Conſequently any two factors in the two extremes may change places; or in the two means.*

For if A : B : : C : D, then AD = BC, and nAD = nBC (Ax. 4); then nA : B : : nC : D (Cor. 1) : : C : $\frac{D}{n}$ (Cor. 1. Prop. 5).

Scholium.

It is ſuppoſed here that two analogous terms are numbers, or at leaſt, that they are repreſented by numbers.

PROP. XIII.

If four quantities are proportional, A : B : : C : D ; *and if the analogous terms be compounded any way by addition or ſubtraction ; ſo that both pairs be ordered alike ; then they will ſtill be proportional.*

If $A : B :: C : D$.

Then $A : A + B :: C : C + D$.

$A : A - B :: C : C - D$.

$A : B - A :: C : D - C$.

$A + B : B :: C + D : D$.

$A - B : B :: C - D : D$.

$B - A : B :: D - C : D$.

$A + B : A - B :: C + D : C - D$.

$A + B : B - A :: C + D : D - C$.

$A : B :: A + C : B + D$.

$A : B :: A - C : B - D$, *&c.* and the reverſe thereof.

For in any caſe, the product of the means is equal to the product of the extremes.

Cor. *When the quantities are compounded after any of the foregoing ways, then it will be,* A : B : : C : D.

PROP. XIV.

If one quantity has the ſame proportion to ſeveral quantities ſeparately ; as a ſecond quantity has to as many others : then the firſt has the ſame proportion to the ſum of the firſt ſet, as the ſecond has to the ſum of the laſt ſet.

If $A : \left\{\begin{matrix} B \\ C \\ D \end{matrix}\right. :: F : \left\{\begin{matrix} G \\ H \\ I \end{matrix}\right.$ *then* $A : B + C + D :: F : G + H + I$.

 For

For $\left.\begin{matrix} B \\ C \\ D \end{matrix}\right\}$: A : : $\left.\begin{matrix} G \\ H \\ I \end{matrix}\right\}$: F (Prop. III), then $\frac{B}{A} = \frac{G}{F}$, $\frac{C}{A} = \frac{H}{F}$, $\frac{D}{A} = \frac{I}{F}$ (Def. 4). Therefore $\frac{B}{A} + \frac{C}{A} + \frac{D}{A}$ or $\frac{B+C+D}{A} = \frac{G}{F} + \frac{H}{F} + \frac{I}{F}$ or $\frac{G+H+I}{F}$ (Ax. 2); therefore B + C + D : A : : G + H + I : F (Def. 4); and A : B + C + D : : F : G + H + I (Prop. III).

Cor. 1. *If one quantity be separately to two quantities; as a second is to two others: the first will be to the difference of the first two; as the second, is to the difference of the last two.*

If A : $\left\{\begin{matrix} B \\ C \end{matrix}\right.$: : F : $\left\{\begin{matrix} G \\ H \end{matrix}\right.$. *Then* A : B — C : : F : G — H.

For then $\frac{B}{A} - \frac{C}{A} = \frac{G}{F} - \frac{H}{F}$ (Ax. 3); and $\frac{B-C}{A} = \frac{G-H}{F}$.

Cor. 2. *The same things supposed as in Cor.* 1, *then* B : C : : G : H.

For $\frac{B}{A} = \frac{G}{F}$, and $\frac{C}{A} = \frac{H}{F}$, whence B : C : : ($\frac{B}{A}$ or $\frac{G}{F}$: $\frac{C}{A}$ or $\frac{H}{F}$: :) G : H (Pr. V. and Cor. 1).

PROP. XV.

If there be two ranks of quantities; and it be, in these two ranks, as the first to the second, so is the first to the second; and as the second to the third, so the second to the third; and so on: then will the first be to the last, as the first to the last, in the two ranks. If A, B, C, D; *and* F, G, H, I, *are two ranks; and it be,* A : B : : F : G, *and* B : C : : G : H, *and* C : D : : H : I; *then* A : D : : F : I.

For $\frac{A}{B} = \frac{F}{G}$, and $\frac{B}{C} = \frac{G}{H}$, and $\frac{C}{D} = \frac{H}{I}$ (Def. 4); therefore $\frac{ABC}{BCD} = \frac{FGH}{GHI}$ (Ax. 4), or $\frac{A}{D} = \frac{F}{I}$; that is, A : D : : F : I.

PROP. XVI.

If two or more rows of quantities are respectively proportional; the like terms are proportional, in any two rows.

If A : B : C : D : : P : Q : R : S. *Then* B : D : : Q : S, *&c.*

Quantities are *respectively proportional,* when in the several rows, the first term is to the first, the second to the second, the third to the third, *&c.* in the same proportion. And *like terms* are those that are alike situated in all the rows; as the third term and the third, the fourth and the fourth, *&c.*

For since B : C : : Q : R, and C : D : : R : S, therefore B : D : : Q : S (Prop. XV); and so of others.

Or thus.

If these are respectively proportional,	A : B : C : D : E : :
	F : G : H : I : K : :
	L : M : N : O : P : :
	Q : R : S : T : V : :
then	A : D : : Q : T; and so of

others.

For

For A : B :: Q : R, and B : C :: R : S, and C : D :: S : T. Therefore A : D :: Q : T. In like manner G : K :: R : V, and A : E :: L : P, and B : E :: R : V, *&c.* all the ways they can be thus compared.

PROP. XVII.

If there be two ſets of quantities; and if it be as the firſt to the ſecond (in the firſt ſet), ſo the laſt but one to the laſt (in the ſecond ſet); and as the ſecond to the third, ſo the laſt but two, to the laſt but one; and ſo on. Then the firſt will be to the laſt (in the firſt ſet), as the firſt to the laſt (in the ſecond ſet).

Firſt ſet A, B, C.
Second ſet F, G, H.

If A : B :: G : H, *and* B : C : F : G, *&c.* *then* A : C :: F : H.

For $\frac{A}{B} = \frac{G}{H}$, and $\frac{B}{C} = \frac{F}{G}$ (Def. 4, 2); therefore $\frac{AB}{BC} = \frac{GF}{HG}$ (Ax. 4), or $\frac{A}{C} = \frac{F}{H}$, and A : C :: F : H.

PROP. XVIII.

If there be four proportional quantities in one rank, and four more in another; and ſeveral ſuch ranks; then the products of the like terms will be proportional.

If A : B :: C : D,
and F : G :: H : I,
and P : Q :: R : S,
then AFP : BGQ :: CHR : DIS.

For $\frac{A}{B} = \frac{C}{D}$, and $\frac{F}{G} = \frac{H}{I}$, and $\frac{P}{Q} = \frac{R}{S}$ (Def. 4), therefore $\frac{AFP}{BGQ} = \frac{CHR}{DIS}$ (Ax. 4), or AFP : BGQ :: CHR : DIS.

Cor. 1. *If* $A : B :: C : D$,
and $B : P :: H : I$,
and $P : Q :: R : S$, *&c.*
then $A : Q :: CHR : DIS$.

For $ABP : BPQ :: A : Q :: CHR : DIS$.

Cor. 2. *The same things supposed with two ranks of proportionals, the quotients of the like terms will be proportional.* $\frac{A}{F} : \frac{B}{G} :: \frac{C}{H} : \frac{D}{I}$.

For $AD = BC$, and $FI = GH$ (Prop. XII); therefore $\frac{AD}{FI} = \frac{BC}{GH}$ (Ax. 5); therefore $\frac{A}{F} : \frac{B}{G} :: \frac{C}{H} : \frac{D}{I}$ (Cor. 1. Prop. XII).

Cor. 3. *The like powers, or the like roots of proportional quantities, will be proportional. If* $A : B :: C : D$, *then* $A^n : B^n :: C^n : D^n$, *and* $\sqrt[n]{A} : \sqrt[n]{B} :: \sqrt[n]{C} : \sqrt[n]{D}$: *n being any number.*

This is plain, by supposing A, F, P all equal; as also B, G, Q; and C, H, R; and also D, I, S.

PROP. XIX.

If between any two quantities proposed, there be interposed any number of terms; the proportion of the first to the last, is compounded of the first to the second, the second to the third, and so on to the last. Suppose A, B, C, D, E, F.

The proportion of A *to* F, *is compounded of* A *to* B, B *to* C, C *to* D, D *to* E, *and* E *to* F.

For $\frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} \times \frac{D}{E} \times \frac{E}{F}$ or $\frac{ABCDE}{BCDEF} = \frac{A}{F}$, all the intermediate terms destroying one another, in the dividend and divisor.

PROP. XX.

In a series of quantities in geometrical progression, A : B : C : D : E : F : G $\div\!\div$; *the product of the extremes is equal to the product of any two means, equally distant from the extremes :* AG = BF = CE, &c.

For since A : B : : F : G (Def. 10); therefore AG = BF (Prop. XII). And since B : C : : E : F; therefore CE = (BF =) AG, and so on.

Cor. *Hence the product of the extremes, is equal to the square of the middle term; when the number of terms is odd.*

PROP. XXI.

If, out of a series of quantities in geometrical progression, there be taken any series of equidistant terms; that series will also be in geometrical progression.

If A : B : C : D : E : F : G : H : I : K : L : M, *in* ∺,
then B : E : H : L *are also* ∺.

For $\frac{B}{C} = \frac{C}{D} = \frac{D}{E} = r$, and $\frac{BCD}{CDE} = r^3 = \frac{B}{E}$ (Ax. 4). Also $\frac{E}{F} = \frac{F}{G} = \frac{G}{H} = r$, and $\frac{EFG}{FGH}$ or $\frac{E}{H} = r^3$; also $\frac{H}{I} = \frac{I}{K} = \frac{K}{L} = r$, and $\frac{HIK}{IKL}$ or $\frac{H}{L} = r^3$, &c. Therefore $\frac{B}{E} = \frac{E}{H} = \frac{H}{L}$ &c. (Ax. 8); and B : E : H : L &c. are ∺ (Def. 10).

PROP. XXII.

If there be a series of quantities in geometrical progression, A : B : C : D : E : F, &c. ∺; *their differences will also be in the same geometrical progression,* A : B : : A — B : B — C : C — D, &c.

For since A : B : : B : C : : C : D, &c. (Def. 10); therefore A : A — B : : B : B — C : : C : C — D &c. (Prop. XIII). And A : B : : A — B : B — C, and B : C : : B — C : C — D (Prop. IV). That is, A : B : C, &c. : : A — B : B — C : C — D, &c.

Cor. *The second, third, fourth differences,* &c. *shall also be in the same geometrical progression.*

PROP.

PROP. XXIII.

If there be a series of quantities in geometrical progression; the ratio of the first, to the second, third, fourth, &c. *is in the simple, duplicate, triplicate,* &c. *ratio of the first to the second, respectively. If* A : B : C : D : E, &c. *then* $\frac{A}{B} = \frac{A}{B}$, $\frac{A}{C} = \frac{AA}{BB}$, $\frac{A}{D} = \frac{A^3}{B^3}$, $\frac{A}{E} = \frac{A^4}{B^4}$, &c.

For $\frac{A}{B} = \frac{B}{C} = \frac{C}{D} = \frac{D}{E}$, &c. (Def. 10). And $\frac{A}{C} = \frac{A}{B} \times \frac{B}{C} = \frac{AA}{BB}$ (Def. 13), $\frac{A}{D} = \frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} = \frac{A^3}{B^3}$; also $\frac{A}{E} = \frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} \times \frac{D}{E} = \frac{A^4}{B^4}$; &c.

PROP. XXIV.

If A, B, C, D, E, &c. *be a set of quantities in geometrical progression, whose differences are infinitely small; and n any number; then it will be,* $A^n : A^n - B^n :: A : n \times \overline{A - B}$.

Since the differences are infinitely small, they will be (nearly) equal, $A - B = B - C = C - D$, &c. and $A - C = \overline{A - B} + \overline{B - C} = 2 \times \overline{A - B}$; $A - D = 3 \times \overline{A - B}$; $A - E = 4 \times \overline{A - B}$, &c. But $A^2 : B^2 :: A : C$, and $A^3 : B^3 :: A : D$, &c. (Prop. XXIII); then

$$A^2 : A^2 - B^2 :: A : A - C = 2 \times \overline{A - B} \text{ (Pr. XIII)}$$

also $A^3 : A^3 - B^3 :: A : A - D = 3 \times \overline{A - B}$.

and $A^n : A^n - B^n :: A : n \times \overline{A - B}$.

PROP.

PROP. XXV.

In a rank of quantities in geometrical progression, A : B : C : D : E, *whose number is n; and the ratio* $r = \frac{A}{B}$; *the last term* (E) $= \frac{A}{r^{n-1}}$ or $\overline{\frac{B}{A}}\Big|^{n-1} \times A$.

For $\frac{A}{B} = r$, or $A = Br$, $B = Cr$, $C = Dr$, $D = Er$.

And $A = Br = Crr = Dr^3 = Er^4$.

Therefore $B = \frac{A}{r}$ the 2d term.

$C = \frac{A}{rr}$ the 3d term.

$D = \frac{A}{r^3}$ the 4th term.

$E = \frac{A}{r^4}$ the 5th term.

And in general the n^{th} term $= \frac{A}{r^{n-1}}$.

PROP. XXVI.

In a rank of quantities in geometrical progression, A : B : C : D : E, *whose number is n, and common ratio* $r = \frac{A}{B}$; *the sum of all the terms is,* $\frac{AA - BE}{A - B} = \frac{BE - AA}{B - A}$.

For A : B :: B : C :: C : D :: D : E (Def. 10). And A : B :: (r : 1 ::) $A + B + C + D : B + C + D + E$, (Prop. X); that is, (putting S = sum), A : B :: S — E : S — A. Therefore $SA - AA = BS - BE$ (Prop. XII); and $SA - SB = AA - BE$, or $SB - AS = BE - AA$ (Ax. 2, 3); therefore $S = \frac{AA - BE}{A - B}$ or $\frac{BE - AA}{B - A}$ (Ax. 5).

Cor.

Cor. 1. *The sum of the terms* $= A + \frac{A - E}{A - B}B$, or $A + \frac{E - A}{B - A}B$.

For $A + \frac{A - E}{A - B}B = \frac{AA - AB + AB - BE}{A - B} = \frac{AA - BE}{A - B}$, &c.

Cor. 2. *In a decreasing geometrical progression, the sum of all the terms* $= \frac{rA - E}{r - 1}$.

For since $r : 1 :: S - E : S - A$. Therefore $S - E = rS - rA$ (Prop. XII); and $rS - S = rA - E$ (Ax. 2, 3); whence $S = \frac{rA - E}{r - 1}$ (Ax. 5).

Cor. 3. *In an increasing geometrical progression; put* $R = \frac{B}{A}$, *then the sum of the terms* $= \frac{R^n - 1}{R - 1}A$.

For $B = RA$, $C = RB = R^2A$, $D = RC = R^3A$, $E = rD = r^4A$, or $E = r^{n-1}A$. But $1 : R :: S - E : S - A$, and $S - A = RS - RE$ (Prop. XII), and $RS - S = RE - A = r^nA - A$ (Ax. 2, 3); whence $S = \frac{r^nA - A}{r - 1}$ (Ax. 5).

PROP.

PROP. XXVII.

In an infinite decreasing geometrical progression, A : B : C : D : E ∺ &c. *Put the ratio* $\frac{A}{B} = \frac{m}{n}$; *then the sum of all the terms* ad infinitum $= \frac{AA}{A-B}$ *or* $\frac{mA}{m-n}$.

For the sum $= \frac{AA - BE}{A - B}$ (Prop. XXVI); but when the progression is infinitely continued, the last term E is *o*, and then the sum becomes $\frac{AA}{A-B}$. Also (by Cor. 2. Prop. XXVI), the sum $= \frac{rA - E}{r - 1}$ becomes $\frac{rA}{r-1} = \frac{\frac{m}{n}A}{\frac{m}{n} - 1} = \frac{mA}{m-n}$.

SECT.

SECT. III.
General Proportions.

Definition and Notation.

IF A, B, C, D, *&c.* be any variable quantities, and a, b, c, d, &c. other values thereof; and if they be ſo dependent on one another, that when A is increaſed or diminiſhed to a; B, C, D, *&c.* become b, c, d, &c.

Then $A \propto B$; ſignifies that A is directly as B, or that $A : a :: B : b$.

Likewiſe $A \propto \frac{1}{C}$, denotes that A is reciprocally as C, or that $A : a :: \frac{1}{C} : \frac{1}{c}$.

Alſo $A \propto \frac{BC}{D}$, ſignifies that A is directly as B and C, and reciprocally as D, or that $A : a :: \frac{BC}{D} : \frac{bc}{d}$.

And if $AB \propto \frac{C}{D}$, the product of A, B is directly as C, and reciprocally as D; or $AB : ab :: \frac{C}{D} : \frac{c}{d}$.

And on the contrary, if $A : a :: B : b$, then $A \propto B$, *&c.*

PROP.

PROP. I.

If one quantity A *is as a second* B; *then, on the contrary, the second* B *is as the first* A. *If* A $\propto$ B, *then* B $\propto$ A.

For $A : a :: B : b$ (Def.).
Therefore $B : b :: A : a$; that is, B $\propto$ A (Def.).

PROP. II.

If one quantity A *is as a second* B, *and the second* B *as the third* C, *and the third* C *as a fourth* D, &c. *then the first* A *is as the last* D. *If* A $\propto$ B $\propto$ C $\propto$ D, *then* A $\propto$ D.

For $A : a :: B : b$,
and $B : b :: C : c$,
and $C : c :: D : d$ (Def.)
therefore $A : a :: D : d$ (Prop. I. Sect. II).
therefore A $\propto$ D (Def.).

Cor. *If one quantity* A *is as a second* B, *and the second* B *reciprocally as a third* C. *Then the first* A *is reciprocally as the third* C. *If* A $\propto$ B $\propto \frac{1}{C}$, *then* A $\propto \frac{1}{C}$.

For $A : a :: B : b : \frac{1}{C} : \frac{1}{c}$; and A $\propto \frac{1}{C}$ (Def.).

PROP.

PROP. III.

If one quantity A *be as a second* B, *and also as a third* C; *then the first* A *will be as the sum or difference of the second and third,* C *and* D. *If* $A \propto B \propto C$, *then* $A \propto B + C$, *or* $A \propto B - C$.

For $A : a :: B : b :: C : c$. Therefore $A : a :: B + C : b + c$, or $A : a :: B - C : b - c$ (Prop. X. Sect. II). And $A \propto B \pm C$.

PROP. IV.

Either side of a general proportion, may be multiplied or divided by any given quantity. If $A \propto B$, *then* $A \propto nB$, *or* $A \propto \frac{B}{n}$.

For $A : a :: B : b :: nB : nb$ (Prop. V. Sect. II) $:: \frac{B}{n} : \frac{b}{n}$ (Cor. 1. ibid.).

PROP. V.

If both sides of a general proportion be multiplied or divided by any variable quantity, they will still be proportional. If $A \propto B$, *and* C *a variable quantity, then* $AC \propto BC$.

For $A : a :: B : b$ (Def.). And $CA : ca :: CB : cb$ (Prop. VI. Cor. 1); that is, $CA \propto CB$.

Also $A : a :: B : b$; and $\frac{A}{C} : \frac{a}{c} :: \frac{B}{C} : \frac{b}{c}$ (Cor. 2. Prop. VI); that is, $\frac{A}{C} \propto \frac{B}{C}$.

Cor. 1. *If* $Q \propto BC$, *then* $\frac{Q}{B} \propto C$, *and* $\frac{Q}{BC}$ *is a given quantity, or always the same.*

For $\frac{Q}{BC}$ is as 1, an invariable quantity.

Cor. 2. *If* $A \propto \frac{1}{B}$, *then* $B \propto \frac{1}{A}$.

For $AB \propto 1$ (Prop. V), $\frac{AB}{A}$ or $B \propto \frac{1}{A}$ (ibid.).

PROP. VI.

Instead of any quantity in one side of a general proportion, one may substitute any other quantity proportional thereto. If $A \propto BC$, *and* $C \propto D$; *then* $A \propto BD$.

For since $C \propto D$, $BC \propto BD$ (Prop. V); whence $A \propto BD$ (Prop. II).

PROP. VII.

If the two sides of one general proportion, be multiplied or divided by the two sides of another general proportion; they will still be proportional. If $A \propto B$, *and* $C \propto D$; *then* $AC \propto BD$, *and* $\frac{A}{C} \propto \frac{B}{D}$.

For $A : a :: B : b$, and $C : c :: D : d$, therefore $AC : ac :: BD : bd$ (Prop. XVIII. Sect. II); that is, $AC \propto BD$.

And $\frac{A}{C} : \frac{a}{c} :: \frac{B}{D} : \frac{b}{d}$ (ibid. Cor. 1); that is, $\frac{A}{C} \propto \frac{B}{D}$.

Cor. 1. *The equal powers or roots of both sides of any general proportion, will still be proportional. If* $A \propto B$, *then* $A^2 \propto B^2$, $A^3 \propto B^3$, $\sqrt[2]{A} \propto \sqrt[2]{B}$, *&c.*

This is plain by putting $C = A$, and $D = B$, *&c.*

Cor. 2. *If* $A \propto B \propto C$, *then* $AA \propto BC$.

PROP. VIII.

If any quantity Q *be as the product of ſeveral others* A, B, *&c. then if* B, *&c. be given,* $Q \propto A$; *and if* A, &c. *be given,* $Q \propto B$.

For by Prop. IV. ſince $Q \propto AB$, and B given, $Q \propto A$. And if A be given, $Q \propto B$ (ibid.).

Cor. *If any variable quantity* Q *depends on ſeveral others* A, B; *and if* $Q \propto A$, *when* B *is invariable*; *and* $Q \propto B$, *when* A *is invariable*; *then* $Q \propto AB$, *when all are variable.*

PROP. IX.

Any general proportion may be turned into an equation, by multiplying one ſide by a proper homologous quantity.

If $A \propto BC$, *then* $A = n \times BC$. *n being ſome given quantity.*

For ſince $A \propto BC$, therefore $A : a :: BC : bc$ (Def.); and $A \times bc = a \times BC$ (Prop. XII. Sect. II); and $A = \frac{a}{bc} \times BC$, therefore $n = \frac{a}{bc}$ the quantity aſſumed for a multiplier.

Or if $mA = BC$, it will be found that $\frac{bc}{a} \times A = BC$, or that $m = \frac{bc}{a}$.

PROP.

PROP. X. *Problem.*

Any general proportion being given, $A \propto \frac{B^2C}{D}$; *to find the proportion any one has to the rest.*

This is done by help of the foregoing propositions.

Since $A \propto \frac{B^2C}{D}$;

Multiply by D, then $AD \propto B^2C$ (Prop. V).

Divide by A, then $D \propto \frac{B^2C}{A}$ (Prop. V).

Divide ($AD \propto B^2C$) by B^2, and then $C \propto \frac{AD}{B^2}$ (Prop. V).

Divide ($AD \propto B^2C$) by C, and then $B^2 \propto \frac{AD}{C}$ (Prop. V).

Extract the square root, $B \propto \sqrt{\frac{AD}{C}}$.

And the same may be done by assuming a given quantity m, and making $mA \doteq \frac{B^2C}{D}$; and the foregoing process is the same as in the reduction of algebraic equations.

FINIS.

THE
ELEMENTS
OF
GEOMETRY.

IN WHICH,

The principal Propoſitions of EUCLID, ARCHIMEDES, and others, are demonſtrated after the moſt eaſy manner.

To which is added,

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THE PREFACE.

HAVING in the first volume treated of arithmetic, which is one of the main pillars of the mathematics; I come now to geometry, which is the other pillar, on which these sciences are supported. On these two foundations, all the other branches are built; and from them they derive their whole strength and evidence. And these two sciences are essentially different; the former considers numbers, without any regard to extension; the latter considers extension, without any regard to numbers. And both of them treat their particular subjects in the most abstract manner.

Geometry is of so excellent a nature and of such extensive use, that it lays the foundation of all the rules to work by, in the common affairs of life, without which we could do nothing. For instance, the distances of places, or remote objects, and their situation in respect of one another; cannot be had without measuring, and the rules of geometry. The drawing of maps or charts can only be done by geometry. The measuring and dividing of lands, to give every man his due share, cannot be performed, without measuring certain figures, and finding their contents. Houses and towns cannot be built without knowing the figures and dimensions

 thereof,

thereof. Without this art, no place can be fortified, to resist the attacks of an enemy. Tradesmen must be acquainted with the measures of length and capacity. Joiners, masons, &c. *must understand how to form their materials into proper figures, where there will be frequent occasion for parallel and perpendicular lines. And the figures they have perpetually to deal with, are triangles, squares, parallelograms, circles,* &c. *and such solids as pyramds, cones, cubes, prisms, spheres,* &c. *the nature of which can only be known from geometry. The dimensions and areas of plane figures, the contents of solid bodies, cannot be had without it. So that geometry gives life and spirit to all arts.*

Geometry examines the nature of all figures, compares them together, and finds out their properties. It is a key to all the other branches. The elements of plane geometry, are likewise the foundation of the higher geometry, relating to all sorts of curve lines, their nature and properties; and is a necessary introduction to the knowledge of them.

Geometry is a science inexhaustible, and which knows no bounds. In it there is always room left for the discovery of new theorems. It is also a most excellent logic, teaches men how to reason truly, and accustoms the mind to a habit of close and strict reasoning.

The science of geometry is certainly very old; for look as far back as we will, we shall always find men who have been professors and encouragers of geometry, and the value the ancients set upon it, may be known from this famous motto of Plato *set over the door of his academy,* οὐδεὶς ἀγεωμέτρητος εἰσίτω.

εἰσίτω. *Some of the principal among them who ſtudied it were,* Thales, Pythagoras, Plato, Ariſtotle, Euclid, Archimedes, Appollonius, Ptolomy, *and many more. But we are not to ſuppoſe that in theſe ancient times, this ſcience was any thing near the perfection it is now in: but in ſucceeding ages, men of great genius, by their ſtudy and induſtry, by degrees added new improvements; till at laſt it arrived at the pitch we now ſee it. So that we need not wonder that* Euclid, *or even* Archimedes, *have taken round-about methods in demonſtrating many of their propoſitions, which are now done vaſtly ſhorter and clearer. For it cannot be denied, that* Euclid's *elements abound with a great many trifling propoſitions, which are of no other uſe but to demonſtrate, in his way, the propoſitions that follow after. But they are diſpoſed in no proper order or method. For he frequently treats of different ſubjects, promiſcuouſly together, in the ſame place; without any regard to the nature of things, or their connection with one another. And as often, has the ſame ſubject to conſider in different places; which can breed nothing but confuſion. But there are likewiſe a great many propoſitions in the preſent ſyſtem of geometry, which theſe ancient mathematicians knew nothing of; and which are equally uſeful with thoſe of* Euclid.

One method of demonſtration, which Euclid *and the ancients frequently make uſe of, is* reductio ad abſurdum, *which is generally ſhorter than the direct method, and equally certain. For it is an axiom in logic, that* that ſuppoſition muſt needs be

be true, which destroys the contrary supposition. *But though it be equally true, yet it gives not that satisfaction to the mind, which a positive proof gives.*

It is a common practice among geometers, after a proposition is proved, for them to prove the reverse of it. But this in many cases is needless and impertinent. For where the essential property of a subject is found; there, most certainly, you will find that subject, without farther inquiry. For example, when it is proved to be the property of parallel lines, when cut by a third line, to make the alternate angles equal; or the sum of the internal angles equal to two right angles: it is superfluous to prove, that when the alternate angles are equal, or the sum of the internal angles equal to two right ones, that these lines are parallel; because it was proved before to be the absolute right and property of parallel lines. Likewise when it is proved to be the distinguishing property of a right-angled triangle, that the square of the hypothenuse is equal to the sum of the squares of the two sides. It need not be proved, that when these squares are equal, the angle is right. In such cases, there needs, at most, nothing but an illustration, and then this method (reductio ad absurdum) *is very properly applied.*

There are also many propositions in geometry, which are convertible; that is, where the property or predicate may become the subject; and the subject, the predicate, being of equal extent. And here a deal of labour might be saved in demonstrating the proposition both ways. For instance, when the

the two ſides of a triangle are equal, it may be proved, that the two oppoſite angles are equal. Or when the two angles of a triangle are equal, it may be proved, that the oppoſite ſides are equal. But it need not be proved both back and forward. And here can want nothing but the application of the former rule (reductio ad abſurdum), *to illuſtrate the reverſe. But mathematicians had rather prove too much than too little; they had rather have ſomething* ex abundanti, *than be defective. Though for my own part, I have often ſaved myſelf that ſuperfluous labour.*

To give ſome account of the method wherein I have handled this ſubject; it is in ſhort this. The firſt book treats of right lines. The ſecond of triangles. The third of polygons. The fourth of the circle. The fifth of planes. The ſixth of ſolids. The ſeventh of the ſphere. The eighth is geometrical problems. This is the method I have choſen to digeſt theſe things in, as being agreeable to the nature of the ſubject, and the mutual dependance of the ſeveral parts upon one another. The laſt book contains a collection of the moſt uſeful geometrical problems. I have ſpent but little time in demonſtrating them, as moſt of them do not need it, being perſuaded that they who underſtand the elements, will eaſily perceive their evidence, without any more words. They that would ſee more problems of this kind, may conſult the writers of practical geometry.

W. Emerſon.

THE

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THE

ELEMENTS

OF

GEOMETRY.

DEFINITIONS.

1. *GEOMETRY* is a ſcience which teaches and demonſtrates the properties, affections, and meaſures of all ſorts of magnitude. FIG.

2. *Magnitude* is continued quantity, or any thing that is extended; as a line, ſurface, or ſolid.

3. A *point* is that which has no parts.

4. A *line* is a length without breadth or thickneſs.

Cor. *The extremes of a line are points.*

5. A *right line* is that which lies evenly, or in the ſame direction, between two points A, B. A *curve line* continually changes its direction. 1.

Cor. *Hence there can only be one ſpecies of right lines, but there is infinite variety in the ſpecies of curves.*

6. *Parallel lines* are thoſe which are always at the ſame diſtance from one another, as AB, CD. 2.

7. An *angle* is the inclination of two lines, to one another, meeting in a point, called the *angular point.* When it is formed by two right lines, it is a *plain angle,* as A; if by curve lines, it is a *curvilineal angle.* 3.

FIG.

8. A *right angle* is that which is made by one right line AB falling upon another CD, and
4. making the angles on each ſide equal, ABC = ABD; ſo that AB does not incline more to one ſide than another: AB is called a *perpendicular*. All other angles are called *oblique angles*.

9. An *obtuſe angle* is greater than a right angle,
5. as R.

10. An *acute angle* is leſs than a right angle, as S.

11. *Contiguous angles*, are thoſe made by one line falling upon another, and joining to one another, as R, S.

12. *Oppoſite angles*, are thoſe made on contrary ſides of two lines interſecting one another, as
6. A, B.

13. A *ſurface* is that magnitude which hath only length and breadth.

Cor. *The extremes or limits of a ſurface are lines.*

14. A *plane* is that ſurface which lies perfectly even between its extremes; or in which, right lines any way drawn, do coincide.

15. A *plain figure*, is a plain ſurface, bounded on all ſides by one or more lines.

16. A *right-lined figure*, is a plain figure, bounded with right lines only.

Cor. *Every right-lined figure has as many angles as ſides.*

17. A *ſolid* is a magnitude extended every way, or which has length, breadth, and depth.

Cor. *The terms or extremes of a ſolid, are ſurfaces.*

18. The *ſquare of a right line* is the ſpace included by four right lines equal to it, ſet perpendicular to one another.

19. The *rectangle of two lines* is the ſpace included by four lines equal to them, ſet perpendicular to one another, the oppoſite ones being equal.

20. Con-

20. *Commenſurable magnitudes*, are ſuch as may be meaſured by one and the ſame meaſure.

21. *Incommenſurable magnitudes*, are ſuch as have no common meaſure.

22. *Rational magnitudes*, are thoſe that are expreſſed by a rational number, or by one that includes not a ſurd.

23. *Irrational magnitudes*, are ſuch as are denoted by a ſurd, as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, *&c.*

AXIOMS or MAXIMS.

1. Things equal to the ſame thing, are equal to one another.

2. The whole is equal to all its parts taken together.

3. If equal things be added to equal things, the wholes will be equal.

4. If equal things be taken away from equal things, the remainde s will be equal.

5. If equal things be equally multiplied, the products will be equal.

6. If equal things be equally divided, the quotients will be equal.

7. All right angles are equal to one another.

8. Thoſe magnitudes are equal, which being applied, exactly agree or coincide with one another.

POSTULATES.

1. Between any two points a right line may be drawn.

2. That a right line or plane may be produced as far as we pleaſe.

3. That a circle may be deſcribed from any center at any diſtance. See Book IV. Def. 1.

4. That any magnitude being given, an equal magnitude may be made.

5. That any magnitude may be ſo often multiplied, as to exceed any magnitude of the ſame kind.

6. That any magnitude may be divided into as many equal parts as we pleaſe.

Explanation of Characters.

$+$ added to, being the ſign of addition.
$-$ ſubtracted from, the ſign of ſubtraction.
$\times$ multiplied by.
$\div$ divided by.
$=$ equal to.
$::$ the mark of proportion.
$\div\!\!\div$ geometrical progreſſion.
$\backsim$ difference.
$\square$ ſquare.
$\sqsubset\!\sqsupset$ rectangle.
$\sqrt{\ }$ ſquare root.
$\sqrt[3]{\ }$ cube root.
A^2 A ſquared; alſo $\overline{AB}^2$ is AB ſquared.
A^3 A cubed; and AB^3 is AB cubed.
$\angle$ an angle.
$\parallel$ parallel.
$\perp$ perdendicular.

Sometimes one letter denotes a line; but if a line is expreſſed by two letters, as AB, then the letters A, B denote the extreme points of that line.

When one letter denotes an angle, it is ſuppoſed to ſtand at the angular point; but if three letters expreſs the angle, the middle one is at the angular point; the other two in the ſides.

When three letters ſtand for a rectangle, as ABC, it ſignifies AB $\times$ BC; where AB, BC are the ſides. Or when four letters ſtand for a rectangle, as AB $\times$ CD; AB and CD are the ſides.

The citations are thus to be underſtood; the firſt number denotes the Prop. the ſecond the Book. When proportion is referred to, it ſignifies geometrical proportion.

BOOK I.

Of Angles, and Right Lines, and their Rectangles.

PROP. I.

If to any point C *in a right line* AB, *ſeveral other right lines* DC, EC *are drawn on the ſame ſide; all the angles formed at the point* C, *taken together, are equal to two right angles,* ACD + DCE + ECB = *two right angles.* FIG. 7.

FOR ſuppoſe PC to be perpendicular to AB, then ſince ACP and PCB = two right angles, (Def. 8); and theſe angles ACD, DCE, ECB occupy the ſame angular ſpace; therefore they are all equal to two right angles (Ax. 2).

Cor. 1. *All the angles made about one point in a plane, being taken together, are equal to four right angles.*

Cor. 2. *If all the angles at* C, *on one ſide of the line* AB, *happen to be equal to two right angles; then* ACB *is a ſtraight line.*

PROP. II.

If two right lines, AB, CD, *cut one another; the oppoſite angles* E *and* G *will be equal.* 8.

For AEC + E = two right angles (Prop. I), and AEC + G = two right angles (ibid.); therefore AEC + E = AEC + G (Ax. 1), and E = G (Ax. 4). After the ſame manner AGC = BGD.

FIG. 8. Cor. *If* AB *is a right line, and* CEB *happen to be equal to* AGD, *or* E *equal to* G; *then* CD *is a right line.*

PROP. III.

9. *A right line* AB, *which is perpendicular to one of two parallels* FH, *is perpendicular to the other* DC.

For ſuppoſe the end HC of the figure CBAH, be raiſed up, and turned over the line AB, ſo that HC may fall towards FD, the line AB remaining fixed. Then ſince the ∠BAH = BAF (Ax. 7), therefore the line AH will fall upon AF, and let the line BC fall on the line B*d*. Draw the line *d*DF perpendicular to HF. Now ſince FH, DC are parallels; therefore the diſtances BA, DF, and *d*F (or CH) are all equal (Def. 6); therefore the points D, *d* muſt coincide; and therefore the line B*d* coincides with BD. Therefore ∠ABC = ABD = a right angle (Def. 8).

Cor. 1. *Hence two lines* FH, DC, *perpendicular to the ſame line* AB, *are parallel.*

Cor. 2. *Hence the ſegments of two parallels, intercepted between two perpendiculars* AB, HC, *are equal,* AH = BC.

For ſince the angles at A, H, B, C are right, therefore the two lines AB, HC, interſecting AH, and being both perpendicular thereto, are parallel (Cor. 1); and therefore AH = BC (Def. 6).

PROP. IV.

10. *If a right line* CG, *interſect two parallels* AD, FH; *the alternate angles,* ABE, *and* BEH, *will be equal.*

Let AE, BH be perpendicular to AD, and FH. Then ſince AE = BH (Def. 6), and AB = EH (Prop. III. Cor. 2), and the angles at A and H right;

right; therefore if the figure EHB be laid upon BAE, the ∠H upon A, and HE upon AB, and consequently HB will fall upon AE; and the whole figure EHB coincides with the figure BAE, and the angle HEB with EBA, and consequently these angles are equal. Likewise the angles DBE and FEB will be equal, being the remainders to two right angles (Ax. 4). FIG. 10.

Cor. 1. *The external angle* CBD, *is equal to the internal angle on the same side* BEH.

For CBD = ABE = BEH (Prop. 2).

Cor. 2. *The two internal angles on the same side are equal to two right angles*; DBE + BEH = *two right angles.*

For EBA = BEH (Prop. IV), and DBE + EBA = two right angles, = DBE + BEH.

Cor. 3. *If the angles* CBD *and* BEH *are equal*; *or* ABE *and* BEH *equal*; *or* DBE + BEH *be equal to two right angles*; *the lines* AD, FH *are parallel.*

For if any angle is greater than is here mentioned, it destroys the parallelism of the lines AD, FH.

PROP. V.

Two lines drawn between two parallels AB, CD, *making equal angles with either of them, will be equal,* AC = BD. 11.

Draw CF, DH perpendicular to FB, then since ∠ACD = BDI; also FCD and HDI right angles (Prop. III), the remainders FCA and HDB are equal; and the angles at F and H being right, and FC = HD (Def. 6); therefore if HD be laid on FC, the line DB will fall on CA, and HB on FA, and B on A; therefore DB = CA.

FIG. Cor. 1. *If the lines* AC *and* BD *are equal, then the angles* ACD *and* BDI *are equal.*

11. For if one angle was greater, it would make the lines AC, BD unequal.

Cor. 2. *The parts intercepted are equal,* AB = CD.

For FA = HB, and adding AH, AH + HB, or AB = FA + AH, or FH = CD (Cor. 2. Prop. III).

Cor. 3. *If two equal and parallel lines* AB, CD, *be joined by two others* AC, BD; *they shall also be equal and parallel.*

PROP. VI.

12. *Right lines* AB, CD, *parallel to the same right line* EF, *are parallel to one another.*

Let GI cut the three lines, then since AB, EF are parallel, AGI = EHI (Cor. 1. Prop. IV); and because EF and CD are parallel, ∠EHI = DIG (Prop. IV). Therefore AGI = DIG (Ax. 1), whence AB, CD, are parallel (Cor. 3. Prop. IV).

PROP. VII.

13. *If two lines* AB, BD, *which cut one another, be parallel to two other lines* EC, CH, *which also cut one another; they shall contain equal angles* ABD = ECH.

For produce EC to intersect BD in F; then by reason of the parallels AB, EF, ∠ABD = EFD (Cor. 1. Prop. IV); and since BD and CH are parallel, EFD = ECH (ibid.); therefore ABD = ECH.

PROP. VIII.

FIG.

Two right lines AF, AB *being given; and one of them* AB *be divided into several parts; the rectangle under the two whole lines, will be equal to all the rectangles contained under the whole line, and the several segments of the other;* ABGF = ADHF + DEIH + EBGI. 14.

For let AF be perpendicular to AB, and DH, EI, BG equal to AF, and also perpendicular to AB. Then AD × AF = rectangle ADHF, and HD × DE, or FA × DE = rectangle DEIH, and IE × EB, or AF × EB = rectangle EBGI (Def. 19); but the sum of these rectangles fill the same space as ABGF, and therefore they are equal (Ax. 8).

Cor. 1. *If both lines be divided into parts, the sum of the rectangles of all the parts, is equal to the rectangle of the wholes.*

Cor. 2. *If the two given lines be equal; the sum of the rectangles under the whole and the parts, is equal to the square of the whole.*

PROP. IX.

If a line AC *be divided into two parts at* B; *the rectangle under the whole, and one of the segments,* AC × BC, *is equal to the rectangle of the segments and the square of the said segment,* AB × BC + BC^2. 15.

Suppose AF, BE, CD all equal to BC, and perpendicular to AC; then the rectangle ACDF = AC × CD = AC × BC (Def. 19); also AB × BC = AB × BE = rectangle ABEF, and BC × CD or BC^2 = BCDF (Def. 18). But ABEF × BCDE fill the rectangle ACDF, and therefore they are equal (Ax. 8).

PROP.

PROP. X.

FIG. 16. *If a right line* AC *be divided into two parts* AB, BC; *the square of the whole line is equal to the squares of both the parts, and twice the rectangle of the parts,* $AC^2 = AB^2 + BC^2 + 2AB \times BC$.

Let AG, BH, CI be equal to AC, and perpendicular thereto, and AD, BE, CF equal to AB; then FI = BC, &c. then ABED is the square of AB (Def. 18), and EFIH is the square of BC; and the figures BF and EG, are the rectangles of BC and BE, and DG and DE; or of AB and BC twice taken (Def. 19). But all these fill the square AI, and therefore are equal to it (Ax. 8).

PROP. XI.

16. *The square of the difference of two lines* AC, BC, *is equal to the sum of their squares, wanting twice their rectangle,* $AB^2 = AC^2 + BC^2 - 2AC \times BC$.

For the square AI contains the square AE, the rectangle CH, and rectangle DH; that is, $AC^2 = AB^2 + CH + DH$; and adding FH, $AC^2 + FH = AB^2 + CH + DI$; that is, $AC^2 + BC^2 = AB^2 + 2ACB$, and AB^2 or $\overline{AC - BC}^2 = AC^2 + BC^2 - 2ACB$.

PROP. XII.

16. *The rectangle of the sum and difference of two lines* AC, AB, *is equal to the difference of their squares,* $\overline{AC + AB} \times BC = AC^2 - AB^2$.

For the difference of the squares AI and AE is the rectangles $CH + HD = \overline{BH + HG} \times BC = \overline{AC + AB} \times BC$.

PROP.

PROP. XIII.

The square of the sum, together with the square of the difference of two lines, is equal to twice the sum of their squares.

Let the lines be A, E. Then
the square of $A+E = A^2+E^2+2AE$ (Prop. X).
the square of $A-E = A^2+E^2-2AE$ (Prop. XI).
then $\overline{A+E}^2 + \overline{A-E}^2 = 2A^2+2E^2$ (Ax. 3).

PROP. XIV.

The difference of the squares, made of the sum and difference of two right lines, is equal to four times their rectangle.

For if A, E be the lines, then

$$\overline{A+E}^2 = A^2+E^2+2AE.$$
$$\overline{A-E}^2 = A^2+E^2-2AE.$$
$$\text{difference} = 4AE.$$

Cor. *The square of the sum is equal to the square of the difference, together with four times their rectangle.*

BOOK II.

Of Triangles.

DEFINITIONS.

1. A *Triangle* is a plain figure bounded by three right lines, called the *ſides* of the triangle.

2. An *equilateral triangle* is that which has three equal ſides.

3. An *equiangular triangle* is that which has three equal angles.

4. An *iſoſceles triangle* is that having two ſides equal.

5. A *right-angled triangle* is that which has a right angle. The ſide oppoſite to the right angle is called the *hypothenuſe*.

6. An *oblique triangle* is that having oblique angles.

7. An *obtuſe angled triangle* has one obtuſe angle.

8. An *acute angled triangle* has three acute angles.

9. A *ſcalenous triangle* has three unequal ſides.

10. *Similar triangles* are thoſe whoſe angles are reſpectively equal, each to each. And *homologous ſides* are thoſe lying between equal angles.

11. *Baſe* of a triangle, is the ſide on which a perpendicular is drawn from the oppoſite angle called the *vertex*; the two ſides, proceeding from the vertex, are called the *legs*.

PROP.

PROP. I.

FIG.

In any triangle ABC, *if one ſide* BC *be drawn out; the external angle* ACD *will be equal to the two internal oppoſite angles* A, B. 17.

Draw CE parallel to AB, then the $\angle$ A $=$ ACE (4. 1); alſo the $\angle$ B $=$ ECD (Cor. 1. ibid.); therefore A $+$ B $=$ ACE $+$ ECD $=$ ACD (Ax. 3).

PROP. II.

In any triangle ABC, *the ſum of the three angles is equal to two right angles*, A $+$ B $+$ C $=$ *two right angles.* 17.

For A $+$ B $=$ ACD (Prop. I), and A $+$ B $+$ C $=$ ACD $+$ ACB (Ax. 3) $=$ two right angles (1. I).

Cor. 1. *If two angles in one triangle, be equal to two angles, in another; the third will alſo be equal to the third.*

Cor. 2. *If one angle of a triangle be a right angle, the ſum of the other two will be equal to a right angle.*

Cor. 3. *There can only be one perpendicular drawn, to any line, from a given point.*

PROP. III.

The angles at the baſe of an iſoſceles triangle, are equal $\angle$ C $=$ B. 18.

For let AD biſect the angle BAC; then if the triangle DAC be laid upon the triangle DAB; then by reaſon of the equal angles at A, and AC $=$ AB, AC will coincide with AB, and C with B, and CD with BD; and therefore $\angle$ ACD $=$ ABD.

Cor. 1. *If the angles* B, C *at the baſe be equal, the ſides* AB, AC *are equal.*

Cor. 2. *An equilateral triangle is alſo equiangular; and the contrary.*

Cor.

FIG. Cor. 3. *The line which is perpendicular to the base of an isosceles triangle, bisects it and the vertical angle.*

Cor. 4. *Only two equal lines can be drawn from a given point to a right line.*

For if AB = AD = AC; then ∠B as well as ∠D = ∠C, which is absurd (Prop. I).

PROP. IV.

In any triangle, the greatest side is opposite to the greatest angle, and the least to the least.

19. Let AC be the greatest side, and suppose AD = AB, then the ∠ADB = ABD (Prop. III), but ADB = DBC + DCB (Prop. I); therefore ADB is greater than C; whence ABD is greater than C, therefore much more is ABC greater than C. After the same manner it is proved, that ABC is greater than A.

And if AB be the least side, C is less than ABC; and may be proved in like manner to be less than A.

PROP. V.

20. *In any triangle* ABC, *the sum of any two sides* BA, AC, *is greater than the third* BC.

Produce the side BA, and let AD = AC, and draw DC; then since ∠ACD = D (Prop. III); therefore BCD is greater than D; and therefore the opposite side BD is greater than BC, that is, BA + AC is greater than BC.

Cor. 1. *A right line is the shortest distance between any two points.*

21. Cor. 2. *The sum of two lines* BD, DC, *drawn from two angles to any point within the triangle, is less than the two sides of the triangle*; BD + DC *is less than* BA + AC, *but contain a greater angle.*

2 For

For drawing BDE, then, in the triangle BAE, BE is lefs than BA + AE (Prop. V), add EC, then BE + EC is lefs than BA + AC. And in the triangle DEC, DC is lefs than DE + EC; add BD, and BD + DC is lefs than BE + EC, and much lefs than BA + AC. FIG.

Alfo $\angle$BDC is greater than DEC, which is greater than A (Prop. I).

PROP. VI.

If two triangles ABC, *abc*, *have two fides and the included angle equal in each; thefe triangles, and their correfpondent parts, fhall be equal.* 22.

For fince the $\angle$A = *a*, and AB = *ab*, alfo AC = *ac*, therefore if A be laid upon *a*, fo that AB fall upon *ab*, then AC will fall upon *ac*, the point B will coincide with *b*, and C with *c*; therefore the whole triangles coincide. Whence the bafe CB = *cb*, $\angle$B = *b*, and C = *c*. And the whole triangles are equal.

Cor. *If two triangles* ABC, *abc*, *have two fides refpectively equal; that which has the greater bafe, has the greater oppofite angle; and the contrary.*

For if the fides CA, BA intercept a greater bafe BC, the angle at A will be fo much the wider or greater; and as the angle increafes, the more of the bafe it intercepts, as is evident.

PROP. VII.

If two triangles ABC *and abc, have two angles and a fide equal, each to each; the remaining parts fhall be equal, and the whole triangles equal.* 22.

For fince two angles are equal, the third will be equal (Cor. 1. Prop. II); therefore if the equal fides BC and *bc* be laid one upon another, then, by reafon of the equal angles B and *b*, C and *c*, the

FIG. the ſides BA and *ba* will coincide, as alſo CA and *ca*, and A will fall on *a*; whence all the parts will be equal (Ax. 8).

PROP. VIII.

If two triangles have all their ſides reſpectively equal; all the angles will be equal, and the wholes equal.

23. For if the baſe of one be laid upon the baſe of the other, the other two ſides will coincide, provided the correſpondent ones lie the ſame way. For if you ſay they don't coincide, let one triangle be ABC, the other ABD: then ſince AB, AC are equal to AB, AD (hyp.), and the angle BAD greater than BAC, therefore BD is greater than BC (Cor. Prop. VI); contrary to the hypotheſis.

Cor. 1. *From two points in a right line, as* A *and* B, *two lines equal to* AC, BC *cannot be drawn to any other point* D.

Cor. 2. *Triangles mutually equilateral, are mutually equiangular.*

PROP. IX.

24. *If in two triangles* ABC, *abc*; *two ſides* AC, CB, *of the one, be equal to ac, cb of the other; and an oppoſite angle* A *equal to the correſpondent oppoſite angle a; and the other oppoſite angles* B, *b*, *either both acute or both obtuſe; the remaining parts of the triangles will be equal.*

For if *cab* be laid upon CAB, ſo that *ca* fall upon CA; then ſince the $\angle a =$ A, *ab* will fall upon ABD. And as *c* falls upon C; *cb* will fall upon either CB or CD (Cor. 4. Prop. III); which here will be CB, as the angle at *b* is obtuſe. Therefore the triangles coincide, and all the parts are equal.

PROP.

PROP. X.

FIG. 25.

Triangles BCA, BCF, *ſtanding upon the ſame baſe, and between the ſame parallels, are equal.*

Let CD be parallel to BA, and BE to CF. Then the triangle CBA = ADC (Prop. VI); for BA = CD (5. 1); and CB = AD (Cor. 2. ibid.), and ∠B = D (4. 1). Therefore the triangle BCA = half of BCDA. For the ſame reaſon BCF = BEF = half of CBEF.

Again, the triangles BAE, CDF are equal, having two ſides and the contained angle equal; add the figure BCDE, and then BCDA = BCFE, and their halves BCA = BCF.

Cor. 1. *Triangles of equal baſes and hights are equal.*

For if their baſes be laid upon one another, the angular points of both (by reaſon of their equal hight) will fall in the ſame parallel; and are therefore equal (Prop. X).

Cor. 2. *Every triangle is equal to half the rectangle of its baſe and hight.*

For ſuppoſe CBA to be a right angle, then it was proved that the triangle CBA is half of the rectangle CBAD; and CBF (equal to it), is therefore equal to half that rectangle.

PROP. XI.

Triangles ABC, ABD, *of the ſame hight, are in proportion to one another as their baſes* BC, *and* BD. 26.

Divide BC into any number of equal parts BF, FG, GH, HC; and BD into ſome number of the ſame equal parts, BI, IK, KD. The triangles ABF, AFG, *&c.* and ABI, AIK, *&c.* are all equal (Cor. 1. Prop. X); and the triangle ABC contains

FIG. 26. ABF as oft as BC contains BF; alſo ABD contains ABI or ABF as oft as BD contains BI or BF; whence ABF : BF : : ABC : BC : : ABD : BD (Def. 4. Proportion and Cor. 2. Prop. XIV. ibid.).

Cor. 1. *Hence triangles are to one another as their baſes and altitudes.*

It follows from this Propoſition, and Cor. 2. Prop. X. therefore,

Cor. 2. *Triangles of equal baſes, are as their hights.*

PROP. XII.

27. *If a line* DE *be drawn parallel to one ſide* BC, *of a triangle; the ſegments of the other ſides will be proportional;* AD : DB : : AE : EC.

For draw BE, DC; then the triangle DEB = triangle DEC (Prop. X); and triangle ADE : BDE : : AD : BD (Prop. XI); and triangle ADE : CDE : : AE : CE (ibid.); therefore AD : DB : : AE : EC (Prop. I. Proportion).

Cor. 1. *If the ſegments be proportional,* AD : DB : : AE : EC; *then the line* DE *is parallel to the ſide* BC.

For if theſe lines were not parallel, the triangles DEB and DEC would not be equal (Cor. 2. Prop. X); and the ſegments would not be proportional.

Cor. 2. *If ſeveral lines be drawn parallel to one ſide of a triangle, all the ſegments will be proportional.*

Cor. 3. *A line, drawn parallel to any ſide of a triangle; cuts off a triangle ſimilar to the whole.*

For ∠D = B, and ∠E = C (Cor. 1. Prop. IV. I); therefore they are ſimilar (Def. 10).

Cor. 4. *The whole ſides are as the ſegments;* AB : DB : : AC : EC.

For

For it is AD : DB :: AE : EC (Prop. XII), FIG.
whence AD + DB (AB) : DB :: AE + EC (AC) 27.
: EC (Prop. XIII. Proportion).

PROP. XIII.

In similar triangles, the homologous sides are proportional; AB : AC :: DE : DF. 28.

In the longer side AC make A*f* = DF, the longer side. And in the shorter side AB, make the shorter side DE = A*e*; and draw *ef*; then the ∠A being supposed = to D, and the comprehending sides equal, ∠A*ef* = E, and A*fe* = F (Prop. VI). Therefore A*ef* = B, and A*fe* = C; consequently *ef* is parallel to BC (Cor. 1. Prop. 4. I); therefore AB : *e*B :: AC : *f*C (Cor. 4. Prop. XII); and AB : AB — *e*B (A*e*) :: AC : AC — *f*C (A*f*), Prop. XIII. Proportion). That is, AB : DE :: AC : DF, or AB : AC :: DE : DF (Prop. IV. Proportion).

And if a triangle was made at the ∠C equal to DFE; it will appear the same way, that AC : CB :: DF : FE. Whence AB : CB :: DE : EF (Prop. XV. Proportion).

Cor. *A line* AE *drawn from the opposite angle* A, *cuts two parallel lines proportionally*; BE : EC :: DI : IF. 29.

For BE : DI :: AE : AI :: EC : IF.

PROP. XIV.

If two triangles have one angle equal to one, and the sides about the equal angles proportional; these triangles are similar. 28.

For let ∠D = A, and let the triangle DEF be laid upon ABC; then, by reason of the equal angles, the sides DE, DF will fall upon AB, AC, the points E, F upon *e* and *f*. Then since DE (A*e*) : DF (A*f*) :: AB : AC, or A*e* : AB :: A*f* : AC, therefore A*e* :

FIG. *eB* : : *Af* : *fC* (Prop. XIII. Proportion) ; whence *ef*
28. is parallel to BC, (Cor. 1. Prop. XII) ; and ∠*e* or E = B, as alſo *f* or F = C (Cor. 1. Prop. IV. I). Whence the triangles DEF, ABC are ſimilar (Def. 10).

PROP. XV.

30. *If two triangles have all their ſides reſpectively proportional, theſe triangles are ſimilar*; AB : DE : : BC : EF : : AC : DF.

Let the ∠FEG = B, and EFG = C, then G = A (Cor. 1. Prop. II) ; whence GE : EF : : AB : BC (Prop. XIII) : : DE : EF (hyp.); therefore GE = DE, (Ax. 7. Proportion). Likewiſe GF : EF : : AC : BC : : DF : EF ; therefore GF = DF (Ax. 7. Proportion). Whence the triangles DEF, GEF have all their ſides reſpectively equal ; and are therefore equiangular ; therefore G = D = A, DEF = GEF = B, and GFE = DFE = C.

PROP. XVI.

If two triangles have one angle in each, equal ; *and the ſides about the ſecond angles proportional* ; *and the third angles both of one kind, acute or obtuſe* ; *theſe triangles are ſimilar.*

31. Let ∠A = D, and AB : BC : : DE : EF. Make ∠ABG = DEF, then ∠G = F (Cor. 1. Prop. II.); whence AB : BG : : DE : EF (Prop. XIII.) : : AB : BC, therefore BG = BC, and BCG is an iſoſceles triangle, and AGB is obtuſe, of the ſame kind with DFE ; and ACB is acute, the ſame as DIE ; whence the angles F and G, or I and C, muſt be of the ſame kind, to have the triangles ſimilar.

SCHOLIUM.

This does not always hold good, if the angles B and E are required to be of the ſame kind, inſtead of

of G and F. For if ABC be acute, ABG will alſo be acute; but ABG is not ſimilar to DEI, nor ABC to DEF; though ABC be ſimilar to DEI, and ABG to DEF. FIG. 31.

PROP. XVII.

Equal triangles, that have one angle equal, have the ſides about the equal angles reciprocally proportional.

Let the oppoſite angles at B be the equal angles, and ABC, DBE, the two equal triangles; then AB : BE : : DB : BC (hyp.). 32.

Draw CE, then AB : BE : : triangle ABC or DBE : triangle CBE (Prop. XI) : : DB : BC.

Cor. 1. *Thoſe triangles are equal, that have the ſides about the equal angles, reciprocally proportional.*

For triangle ABC : CBE : : AB : BE (Prop. XI) : : DB : BC (hyp.) : : triangle DBE : CBE; therefore triangle ABC = DBE (Ax. 7. Proportion).

Cor. 2. *Equal triangles have their baſes and hights reciprocally proportional.*

For each triangle is equal to a right-angled triangle of the ſame baſe and hight (Prop. X); and then the ſides about the right angles, are reciprocally proportional (Prop. XVII).

PROP. XVIII.

Like triangles ABC *and* DEF *are in the duplicate ratio, or as the ſquares of, their homologous ſides,* BC, EF. 33.

Let there be taken BG, ſo that BC : EF : : EF : BG, and draw AG. Then ſince AB : DE : : BC : EF (Prop. XIII) : : EF : BG (Conſtruct.); therefore the triangle ABG = DEF. But ABC : ABG or DEF : : BC : BG (Prop. XI) : : BC^2 : EF^2 (Prop. ~~XXIII~~. Proportion).

FIG.

PROP. XIX.

34. *Triangles that have one angle equal to one, are to one another in the complicate ratio of the ſides about the equal angles*; ABC : EBD : : AB × BC : EB × BD.

Draw CE, then CD, AE being ſtraight lines, the angles at B are equal (Prop. II. 1). Then triangle ABC : CBE : : AB : BE (Prop. XI), and CBE : EBD : : CB : BD (ibid.); therefore ABC : EBD : : AB × CB : BE × BD (Cor. 1. Prop. XVIII. Proportion).

PROP. XX.

35. *In a right-angled triangle* BAC, *if a perpendicular be let fall from the right angle upon the hypothenuſe, it will divide it into two triangles ſimilar to one another and to the whole*, ABD, ADC.

For in the triangles ABD, ABC, the angle B is common to both, and angles D and BAC are right ones; therefore the remaining angles BAD and BCA are equal; therefore the triangles ABD and ABC are ſimilar.

Again, in the triangles ACD and ACB, ∠C is common, ∠D = CAB, and therefore ∠DAC = B, therefore ACD and ABC are ſimilar; and conſequently ABD and ADC.

Cor. 1. *The rectangle of the hypothenuſe and either ſegment is equal to the ſquare of the adjoining ſide.*

For BD : BA : : BA : BC (Prop. XIII), and CD : CA : : CA : CB (ibid.); whence BD × BC = BA^2, and CD × CB = CA^2 (Prop. XII. Proportion).

Cor. 2. *The rectangle of the hypothenuſe and perpendicular, is equal to the rectangle of the legs.*

For BC : AB : : AC : AD (Prop. XIII), and AB × AC = BC × AD (Prop. XII. Proportion).

Cor.

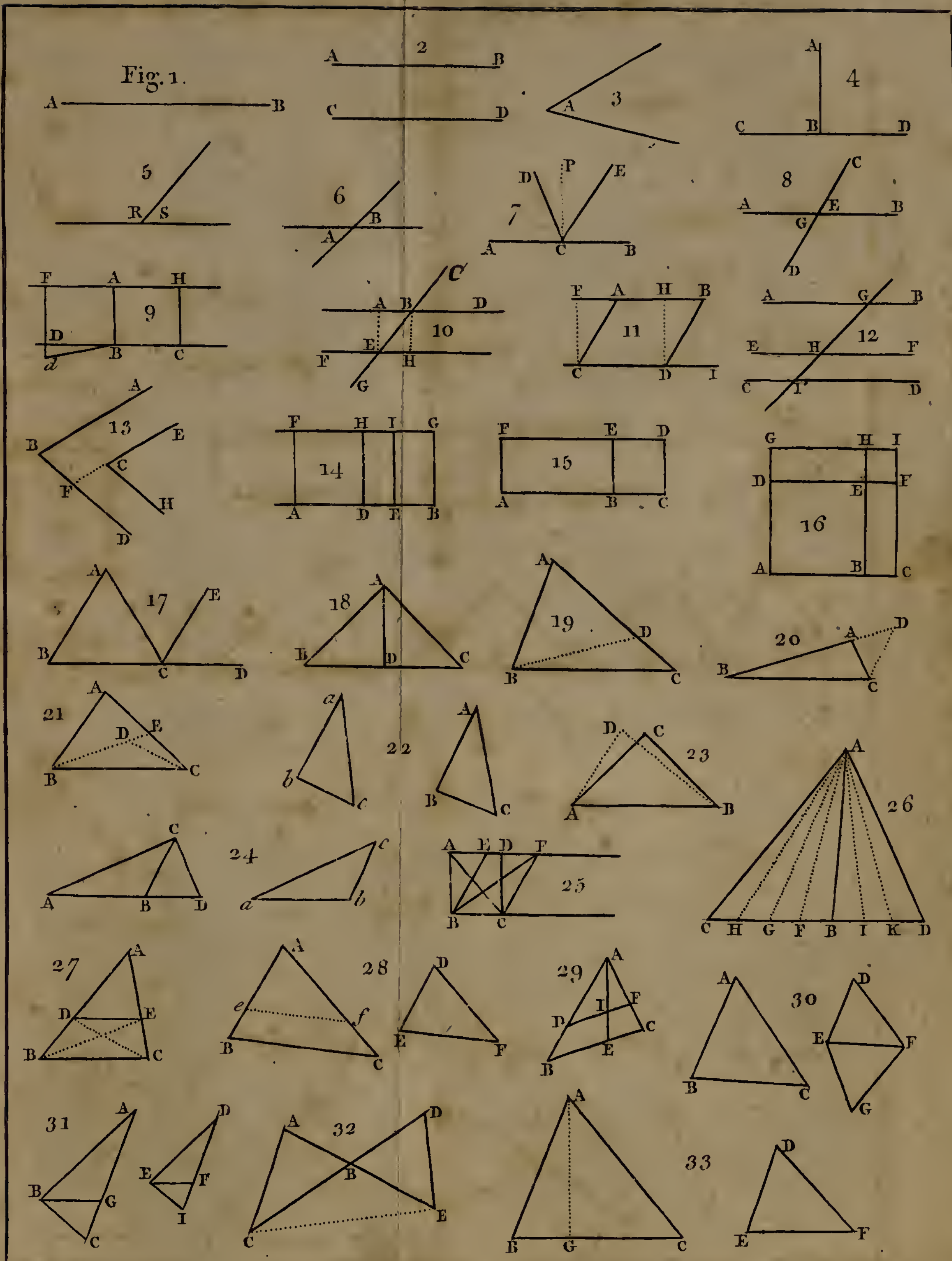
Fig. 1.
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Cor. 3. *The perpendicular is a mean proportional between the segments of the hypothenuse.* FIG. 35.

For BD : DA : DC, and BD $\times$ DC $=$ DA².

Cor. 4. *The segments of the hypothenuse are as the squares of the adjoining sides.*

For by this Prop. BD : DA : : BA : AC (Prop. XIII), and BD² : DA² : : BA² : AC² (Cor. 3. Prop. XVIII. Proportion). And by Cor. 3. (and Prop. XXIII. Proportion) BD : DC : : BD² : DA² : : BA² : AC².

Cor. 5. *As the perpendicular, to the hypothenuse; so the rectangle of the segments, to the rectangle of the legs.*

For AD : AB : : CD : CA, by the sim. triangles BAD, DAC.

And BA : BC : : BD : BA by the sim. triangles BAC, BAD.

Therefore AD : BC : : BDC : BAC (Cor. 1. Prop. XVIII. Proportion).

Cor. 6. *The distance of the right angle, from the middle of the hypothenuse, is equal to half the hypothenuse.*

For let B*o* $=$ *o*C, and draw *on*, *or* parallel to AC, AB; and draw A*o*. Then B*n* $=$ *n*A, and C*r* $=$ *r*A (Prop. XII); and the angles at *n* and *r* are right (Cor. 1. Prop. IV. I). Then the triangles B*on*, A*on*, as also the triangles C*or*, A*or*, have two sides, and the included angle, equal; therefore B*o* $=$ A*o* $=$ C*o* (Prop. VI).

PROP. XXI.

In a right-angled triangle BAC, *the square of the hypothenuse* BC, *is equal to the sum of the squares of the two sides*, BA, AC. 36.

For

FIG. 36. For let BG be the ſquare deſcribed on BC, and draw ADF perpendicular to BC, or parallel to CG or BE. Then BA^2 = rectangle of BD and BC or BE (Cor. 1. Prop. XX), = rectangle BF. Alſo the ſquare of AC = rectangle of CD and CB = rectangle CF (ibid.): but rectangle BF + CF = ſquare BG (Ax. 8); therefore BG or the ſquare of $BC = BA^2 + AC^2$.

Cor. 1. *The ſquare of either ſide is equal to the difference between the ſquares of the hypothenuſe and the other ſide;* $BA^2 = BC^2 - AC^2$, *and* CA^2 $BC^2 - BA^2$.

Cor. 2. *The rectangle of the ſum and difference of the hypothenuſe and one of the ſides, is equal to the ſquare of the other ſide.*

For $BA^2 = BC^2 - AC^2$ (Cor. 1) $= \overline{BC + AC} \times \overline{BC - AC}$ (Prop. XII. I).

Cor. 3. *If the ſquare of one ſide of a triangle be equal to the ſum of the ſquares of the other two ſides; then the angle comprehended by them is a right angle.*

For if it was greater or leſs than a right angle, the oppoſite ſide would be greater or leſs than the hypothenuſe of a right-angled triangle (Cor. Prop. VI); and its ſquare greater or leſs than the ſquares of the other ſides.

Cor. 4. *A perpendicular* CA *is the neareſt diſtance of a point* C, *from a right line* BA.

Cor. 5. *In any triangle* ACB, *if a perpendicular be let fall from the oppoſite angle* A, *on the baſe* CB. *The difference of the ſquares of the ſides, is equal to the difference of the ſquares of the ſegments,* $AB^2 - AC^2 = BD^2 - CD^2$.

For $AB^2 - BD^2 = AD^2 = AC^2 - CD^2$ (Cor. 1. Prop. XXI). And $AB^2 - AC^2 = BD^2 - CD^2$ (Ax. 3, 4).

PROP.

PROP. XXII.

FIG.

In an obtuse angled triangle ABC, *if a perpendicular be let fall upon the base; or one side adjoining to the obtuse angle* B; *then the square of the side opposite to that obtuse angle is equal to the sum of the squares of the two lesser sides, together with twice the rectangle of the base and the distance of the perpendicular from the obtuse angle:* $AC^2 = AB^2 + CB^2 + 2CBD$. 37.

For $AC^2 = AD^2 + CD^2$ (Prop. XXI) $= AD^2 + CB^2 + BD^2 + 2CBD$ (10. 1) $= AB^2 + CB^2 + 2CBD$ (Prop. XXI).

Cor. *The distance of the perpendicular from the obtuse angle,* $BD = \frac{AC^2 - AB^2 - CB^2}{2CB}$.

PROP. XXIII.

If a perpendicular be let fall upon the base, or side adjoining to an acute angle B, *of any triangle. Then,* 38. 39.

The square of the side opposite to that acute angle, together with twice the rectangle, of the base, and the distance of the perpendicular from the acute angle; is equal to the sum of the squares of the two other sides: $AC^2 + 2CBD = AB^2 + BC^2$.

For $AC^2 = AD^2 + DC^2$ (Prop. XXI) $= AD^2 + BC^2 + BD^2 - 2BD \times BC$ (Prop. XI. I) $= AB^2 + BC^2 - 2CBD$ (Prop. XXI). And $AC^2 + 2CBD = AB^2 + BC^2$ (Ax. 3).

Cor. *The distance of the perpendicular from the acute angle* B *is* $= \frac{AB^2 + BC^2 - AC^2}{2CB}$.

PROP. XXIV.

In any triangle ABC, *let fall a perpendicular* AD *on the base* BC, *and make* DF = DB. *Then* 40. 41.

As the base, CB :

to sum of the sides, AC + AB : :

So

FIG. So *difference of the sides*, AC — AB :

40. *to difference of the segments of the base*,

41. *or the alternate base* } CF.

For $CA^2 - AB^2 = CD^2 - DB^2$ (Cor. 5. Prop. XXI); that is,

$\overline{CA + AB} \times \overline{CA - AB} = CF \times CB$ (Prop. XII. I). whence CB : CA + AB : : CA — AB : CF (Cor. 1. Prop. XII. Proportion).

Cor. *The difference of the squares of the sides, is equal to twice the rectangle of the base, and the distance of the perpendicular from the middle of the base*, $CA^2 - AB^2 = 2CB \times oD$.

For if $Co = oB$, then $CA^2 - AB^2 = CF \times CB = \frac{1}{2}CF \times 2CB$; but $\frac{1}{2}CF = Do$; for (Fig. 40) $CF = 2Bo - FB$, and $\frac{1}{2}CF = Bo - BD = Do$. And (Fig. 41) $CF = 2Bo + FB$, and $\frac{1}{2}CF = Bo + BD = Do$.

PROP. XXV.

42. *If an angle* A *of a triangle* BAC *be bisected by a right line* AD, *which cuts the base; the segments of the base will be proportional to the adjoining sides of the triangle*; BD : DC : : AB : AC.

Produce BA, and make AE = AC, and draw the line CE; because AE = AC, the ∠ACE = E (Prop. III) = $\frac{1}{2}$BAC (Prop. I) = BAD (hyp.). Therefore DA, CE are parallels (Cor. 3. Prop. IV). Therefore BA : AE or AC : : BD : DC (Prop. XII).

Cor. 1. *If the sides be as the segments of the base; the line* AD, *bisects the angle* A.

For since BA : AC or AE : : BD : DC, DA and CE are parallels (Cor. 1. Prop. XII); and BAD = ∠E, and DAC = ACE = E (Prop. III). Whence BAD = DAC, and A is bisected by AD.

43. Cor. 2. *If a line bisecting the vertical angle of a triangle cuts the base, it will be*

As the sum of the sides, BA + AC : FIG.
to their difference, BA — AC : : 43.
So the base, BC :
to difference of the segments BD — DC.

For BA : AC : : BD : DC (Prop. XXV), and BA + AC : BA — AC : : BD + DC (BC) : BD — DC or 2DO (Prop. XIII. Proportion); where O is the middle point of the base BC.

PROP. XXVI.

If an angle A *of a triangle* ABC, *be bisected by a right line* AD, *which cuts the base; the square of the bisecting line, together with the rectangle of the segments, is equal to the rectangle of the sides*; AD^2 + BDC = BAC. 43.

Produce AD and make the ∠DBP = DAC. Then the three triangles CDA, PDB, and PBA are similar. For AD = PBD = PAB, CDA = PDB (2. I), whence C = P, and ADC = ABP (Cor. 1. Prop. II). Therefore CD : DA : : PD : BD (Prop. XIII), whence DA × PD = CD × BD (12. Proportion). Again, CA : DA : : AP or AD + DP : AB (Prop. XIII), therefore CA × AB = AD^2 + DA × DP (12. Proportion) = AD^2 + CD × BD (Ax. 3).

PROP. XXVII.

In an isosceles triangle ABC, *if a line be drawn from the vertex to cut the base; the square of that line, together with the rectangle of the segments of the base, is equal to the square of the side*; BE^2 + AEC = BA^2. 44.

Let BD be perpendicular to the base, then $BA^2 = BD^2 + AD^2$ (Prop. XXI) = $BD^2 + \overline{AE + ED}^2$ = $BD^2 + AE^2 + ED^2$ + 2AED (Prop. X. I.) = $BE^2 + AE^2$ + 2AED (Prop. XXII)
= BE^2

FIG. $= BE^2 + AE \times \overline{AE + 2ED}$ (Prop. IX. I)
44. $= BE^2 + AE \times EC$, because $AE + 2ED = EC$. For $2AE + 2ED = AC$, therefore taking away AE, $AE + 2ED = EC$.

PROP. XXVIII.

45. *In any triangle* BAC, *if a line* AD *be drawn from the vertex to the middle of the base. The sum of the squares of the sides, is equal to twice the square of half the base, together with twice the square of the line that bisects the base*; $AB^2 + AC^2 = 2AD^2 + 2DC^2$.

For $AC^2 + 2CDP = AD^2 + DC^2$ (Prop. XXIII), and $DC = DB$ (hyp.),
therefore $AC^2 = AD^2 + DC^2 - 2CDP$ (Ax. 4);
and $AB^2 = AD^2 + DB^2 + 2CDP$ (Prop. 22),
therefore $AB^2 + AC^2 = 2AD^2 + 2DC^2$ (Ax. 3).

Cor. $AB^2 - AC^2 = (4CDP =) 2BC \times DP$.

PROP. XXIX.

46. *If through any point* E, *within a triangle* ABC, *three lines* TQ, VR, PS, *be drawn parallel to the three sides of the triangle; the product or solid made by the alternate segments of these lines, will be equal.* $TE \times PE \times RE = QE \times SE \times VE$.

The triangles TEV, PEQ, SER, and ABC are all similar (7. I), whence

$TE : VE :: AC : BC$ (Prop. XIII).
$PE : QE :: AB : AC$.
$RE : SE :: BC : AB$.

whence $TE \times PE \times RE : VE \times QE \times SE :: AC \times AB \times BC : BC \times AC \times AB$ (Prop. XVIII. Proportion). But the two last terms are equal, therefore $TE \times PE \times RE = VE \times QE \times SE$ (Prop. II. Proportion).

PROP.

PROP. XXX.

FIG. 46.

If three lines AF, BG, CD, *be drawn through any point* E, *within a triangle* ABC, *to the opposite sides; the products of the alternate segments of the sides are equal; that is,* $AG \times CF \times BD = CG \times BF \times AD$.

For drawing TQ, VR, PS parallel to the sides of the triangle, then

AG : GC : : TE : QE (Cor. Prop. XIII).
CF : BF : : RE : VE.
BD : AD : : PE : SE.

whence $AG \times CF \times BD : GC \times BF \times AD :: TE \times RE \times PE : QE \times VE \times SE$ (Prop. XVIII. Proportion), but the two last are equal (Prop. XXIX); therefore $AG \times CF \times BD = GC \times BF \times AD$ (Prop. II. Proportion).

PROP. XXXI.

Three lines drawn from the three angles of a triangle to the middle of the opposite sides, all meet in one point.

Let BD, AE bisect the opposite sides AC, BC; and through the point of intersection G, draw CGK, and EL, DI parallel to it. 47.

Now since BE = EC, and AD = DC, we have BL = LK, and AI = IK (Prop. XII). Also since BE = $\frac{1}{2}$BC, and AD = $\frac{1}{2}$AC, it will be EH = $\frac{1}{2}$CG = DF (Prop. XIII). Therefore the triangles DGF, HGE, having all the angles equal (4. I), are similar and equal (Prop. VII); whence FG = GE, and consequently IK = KL (Cor. 2. Prop. XII), therefore AI = IK = KL = LB = $\frac{1}{4}$AB. And AK = KB. And therefore if the line CK be drawn through the middle point K, it will pass through G; otherwise the line passing through G, would make AK greater or lesser than KB. This may also be demonstrated from Prop. XXX.

Cor.

FIG. Cor. *Hence the distance of the point of intersection*
47. G, *from any angle, is twice the distance from the opposite side*, BG = 2GD, *&c.*

For since BK = 2KI, and AK = 2KL, therefore BG = 2GD, and AG = 2GE. Also since DI = DF + FI = 3HL or 3FI, therefore 2FI = DF = GK = EH = $\frac{1}{2}$CG.

PROP. XXXII.

Three perpendicular lines erected on the middle of the three sides of any triangle, all meet in one point.

48. Let E, F be the middle points of AB, CB; FO, EO two perpendiculars. From O draw OD perpendicular to AC. The right-angled triangles COF, BOF are similar and equal, and CO = OB (Prop. VI); also the right-angled triangles BOE, AOE, are similar and equal, whence BO = OA (ibid.); therefore CO = AO; therefore in the isosceles triangle AOC, the perpendicular OD bisects the base AC (Cor. 3. Prop. III): and if it bisects the base, it passes through O.

Cor. *The point of intersection* O, *of the three perpendiculars, will be equally distant from the three angles.*

For the triangles COF, BOF, are similar and equal (Prop. VI), and OB = OC. Also the triangles COD, AOD, are similar and equal (ibid.), and CO = AO = BO.

PROP. XXXIII.

49. *If two right-angled triangles* BID, BED, *be described upon one hypothenuse* BD, *lying on different sides thereof, and the line* EI *drawn to the opposite angles; I say, the angles* DBI *and* DEI *are equal, which stand upon the same side* DI.

Make

Make BC = CD; draw ECF and CI. Then CD, CI, CB, and CE are all equal (Cor. 6. Prop. XX). The external angle ICD = CIB + CBI (Prop. I) = 2CBI (Prop. III). Alſo the external angle ICF = EIC + IEC = 2IEC (ibid.). Alſo FCD = CDE + CED = 2CED (ibid.). Therefore by addition ICF + FCD, that is, ICD = 2AED = 2CBI, and AED = CBI, or IED = IBD. FIG. 49.

PROP. XXXIV.

Three perpendiculars drawn from the three angles of a triangle, upon the oppoſite ſides, all meet in one point.

Let AI, CE be perpendicular to CB, AB; and through the point of interſection D draw BDF; draw CK perpendicular to CA, alſo draw EI. 50.

The oppoſite angles IDC and EDA are equal (2. I), and the angles at E and I are right, therefore the triangles ADE and CDI are ſimilar, whence AD : ED : : CD : DI (Prop. XIII); therefore the triangles ADC, and EDI are ſimilar (Prop. XIV), and angle DEI = DAC = ICK (Prop. XX): But the triangles DBE, DBI are right-angled at E and I, whence ∠DEI = DBI (Prop. XXXIII); therefore DBI or FBC = ICK, and therefore BF is parallel to CK (Cor. 3. Prop. IV), or perpendicular to AC. And if BF be perpendicular to AC, it will paſs through D.

PROP. XXXV.

Three lines biſecting the three angles of a triangle, all meet in one point.

For let CDF and ADE biſect the angles C, A; and through D, the point of interſection, draw BDG. Then BC : CG : : BD : DG : : BA : AG (Prop. XXV); and BC : BA : : CG : AG (Prop. IV. Proportion); whence BDG biſects the angle B (Cor. 51.

FIG. (Cor. 1. Prop. XXV), therefore the line bisecting
51. the ∠B, passes through D.

Cor. 1. *If two lines bisect two angles of a triangle, the point of intersection* D, *is equally distant from the three sides.*

Let Dn, Do, Dp be perpendicular on the three sides. Then the triangles BDn, BDo have one side and all the angles equal, therefore Dn = Do (Prop. VII); also the triangles ADo, ADp, have one side and all the angles equal; therefore Do = Dp (ibid.) = Dn.

Cor. 2. *Segment* Ap + *the opposite side* BC = *half the sum of the sides.*

For half the sum of the sides = Ap + Cn + Bn.

PROP. XXXVI.

52. *If the three angles of a triangle be bisected by the lines* AC, BC, DC, *and any one* BC *continued to the opposite side, and* CP *be drawn perpendicular to that side*, AD; *I say, the angle* ACE = DCP, *or* ACP = DCE.

For since ∠A + B + D = two right angles (Prop. II), therefore CAB + CBA + CDP = a right angle = DCP + CDP (Cor. 2. Prop. II); therefore CAB + CBA or ACE (Prop. I) = DCP.

PROP. XXXVII.

53. *The area of a right-angled triangle* ABC, *is equal to the rectangle under half the perimeter, and its excess above the hypothenuse.*

The perimeter or circumference is the sum of the three sides. Now since the triangle ABC is right-angled at C, the area = $\frac{AC \times CB}{2}$ (Cor. 2. Prop. X); and $AB^2 = AC^2 + CB^2$ (Prop. XXI), or $AC^2 + CB^2 - AB^2 = 0$. Hence four times the

the area $= 2AC \times CB = AC^2 + CB^2 +$ FIG.
$2ACB - AB^2 = \overline{AC + CB}^2 - AB^2$ (10. I) 53.
$= \overline{AC + CB + AB} \times \overline{AC + CB - AB}$ (12. I).

And the area $= \frac{AC+CB+AB}{2} \times \frac{AC+CB-AB}{2}$.

But $\frac{AC+CB-AB}{2} = \frac{AC+CB+AB}{2} - AB$.

Cor. *The area of a right-angled triangle, is equal to the rectangle under the two excesses, of half the perimeter above each side*; $\frac{AC+CB+AB}{2} - BC$, *and* $\frac{AC+CB+AB}{2} - AC$.

For $\frac{AC+CB+AB}{2} - CB = \frac{AB+AC-BC}{2}$,

and $\frac{AC+CB+AB}{2} - AC = \frac{AB+CB-AC}{2}$,

and $\frac{AB + AC - BC}{2} \times \frac{AB + BC - AC}{2} =$

$\frac{AB + \overline{AC-CB}}{2} \times \frac{AB - \overline{AC-CB}}{2} = \frac{AB^2 - \overline{AC-CB}^2}{4}$

$= \frac{AB^2 - AC^2 - CB^2 + 2ACB}{4}$ (Prop. XXI) $=$

$\frac{ACB}{2} =$ area (Cor. 2. Prop. X).

PROP. XXXVIII.

In any triangle ABC; *add the three sides together* 54.
into one sum; and likewise from the sum of every two sides, subtract the third; and you will have three remainders. Then take the product of the said sum, and one of the remainders; and likewise the product of the other two remainders.

Then I say, four times the area of the triangle, is a mean proportional, between these two products.

Take AE, and AF, equal to AC, and draw CF, CE; also draw CD perpendicular to AB. Then

FIG. $AB \times CD =$ twice the area (Cor. 2. Prop. X).
54. And the angle FCE is a right angle; for $AFC = ACF$ (Prop. III), and $AEC = ACE$ (ibid.); therefore $AFC + AEC = ACF + ACE = FCE$ (Ax. 3) $=$ a right angle (Cor. 2. Prop. II). And $AD = \frac{AB^2 + AC^2 - CB^2}{2AB}$ (Cor. Prop. XXIII).

Now $DE = AE - AD = AC - AD$ $= \frac{AC \times 2AB - AB^2 - AC^2 + CB^2}{2AB} =$ $\frac{\overline{CB + AB - AC} \times \overline{CB + AC - AB}}{2AB}$ (11. I).

Also $FD = FE - DE = 2AC - DE = \frac{2AC \times 2AB}{2AB}$ $- DE = \frac{2AC \times 2AB - AC \times 2AB + AB^2 + AC^2 - CB^2}{2AB}$ $= \frac{AB^2 + AC^2 + 2AC \times AB - CB^2}{2AB} =$ $\frac{\overline{AB + AC + BC} \times \overline{AB + AC - BC}}{2AB}$ (12. I); but DC is a mean between DE and DF (Cor. 3. Prop. XX), therefore $DC \times 2AB$ is a mean between $DE \times 2AB$ and $DF \times 2AB$ (Prop. V. Proportion); that is, four times the area of the triangle ABC, is a mean proportional, between $\overline{CB + AB - AC} \times \overline{CB + AC - AB}$, and $\overline{AB + AC + BC} \times \overline{AB + AC - BC}$.

55. Cor. 1. *From half the ſum of the three ſides of any triangle* ABC, *ſubtract each ſide ſeparately. Then take the product of that half ſum and one remainder; and alſo the product of the other two remainders.*

Then I ſay, the area of the triangle is a mean proportional between theſe two products.

For $\frac{CB + AB - AC}{2} \times \frac{CB + AC - AB}{2}$: area ABC : $\frac{AB + AC + BC}{2} \times \frac{AB + AC - BC}{2}$

(Cor.

(Cor. 1. Prop. V. Proportion) are in continual proportion (Prop. XXXVIII). FIG. 55.

But $\frac{CB + AB - AC}{2} = \frac{CB + AB + AC}{2} - AC.$

and $\frac{CB + AC - AB}{2} = \frac{CB + AB + AC}{2} - AB.$

and $\frac{AB + AC - BC}{2} = \frac{CB + AB + AC}{2} - BC.$

therefore, &c.

Cor. 2. *Let* $S = AC + BC$, $D = AC - BC$, 56.
then the area ABC *is a mean proportional between* $\frac{1}{4} \times \overline{SS - AB^2}$, *and* $\frac{1}{4} \times \overline{AB^2 - DD}$.

For $\frac{1}{4} \times \overline{SS - AB^2} = \frac{S + AB}{2} \times \frac{S - AB}{2} = \frac{AC + BC + AB}{2} \times \frac{AC + BC - AB}{2}$, and $\frac{1}{4} \times \overline{AB^2 - DD} = \frac{AB + D}{2} \times \frac{AB - D}{2} = \frac{AB + AC - BC}{2} \times \frac{AB + BC - AC}{2}$, which is the same, as Cor. 1. supposing two terms in the extremes to change places, by Cor. 3. Prop. XII. Proportion.

PROP. XXXIX.

The square of the side of an equilateral triangle, is to the area; as 4 *to* $\sqrt{3}$.

Let CD be perpendicular to AB, then $AD = DB = \frac{1}{2}AB$. Then $CD^2 = CA^2 - AD^2$ (Cor. 1. Prop. XXI) $= AB^2 - \frac{1}{4}AB^2 = \frac{3}{4}AB^2$. And $CD = \sqrt{\frac{3AB^2}{4}} = \frac{AB}{2}\sqrt{3}$. But the area of the triangle is $AB \times \frac{1}{2}CD = AB \times \frac{AB}{4}\sqrt{3}$, and $4 \times \text{area} = AB^2 \times \sqrt{3}$ (Cor. 2. Prop. X); whence $AB^2 : \text{area} :: 4 : \sqrt{3}$. 57.

Cor. *The square of the perpendicular is equal to* $\frac{3}{4}$ *the square of the side*; $CD^2 = \frac{3}{4}CA^2$.

For $CD^2 = CA^2 - AD^2$ (Cor. XXI) $= CA^2 - \frac{1}{4}CA^2 = \frac{3}{4}CA^2$.

BOOK III.

Of Quadrangles and Polygons.

DEFINITIONS.

FIG. 1. A *Quadrangle or quadrilateral*, is a plane figure bounded by four right lines.

58. 2. A *parallelogram* is a quadrangle whose opposite sides are parallel, as AGBH. The line AB drawn to the opposite corners is called the *diameter* or *diagonal*. And if two lines be drawn parallel to the two sides, through any point of the diagonal; they divide it into several others, and then C, D are called *parallelograms about the diameter:* and E, F *the complements:* and the figure EDF a *gnomon*.

3. A *rectangle* is a parallelogram whose sides are perpendicular to one another.

4. A *square* is a rectangle of four equal sides.

59. 5. A *rhombus* is a parallelogram, whose sides are equal, and angles oblique.

58. 6. A *rhomboides* is a parallelogram, whose sides are unequal, and angles oblique.

60. 7. A *trapezoid* is a quadrangle, having only two sides parallel.

61. 8. A *trapezium* is a quadrangle, that has no two sides parallel.

62. 9. A *polygon* is a plane figure enclosed by many right lines. If all the sides and angles are equal, it is called a *regular polygon*, and denominated according to the number of sides, as a *pentagon* 5 sides, a *hexagon* 6, a *heptagon* 7, &c.

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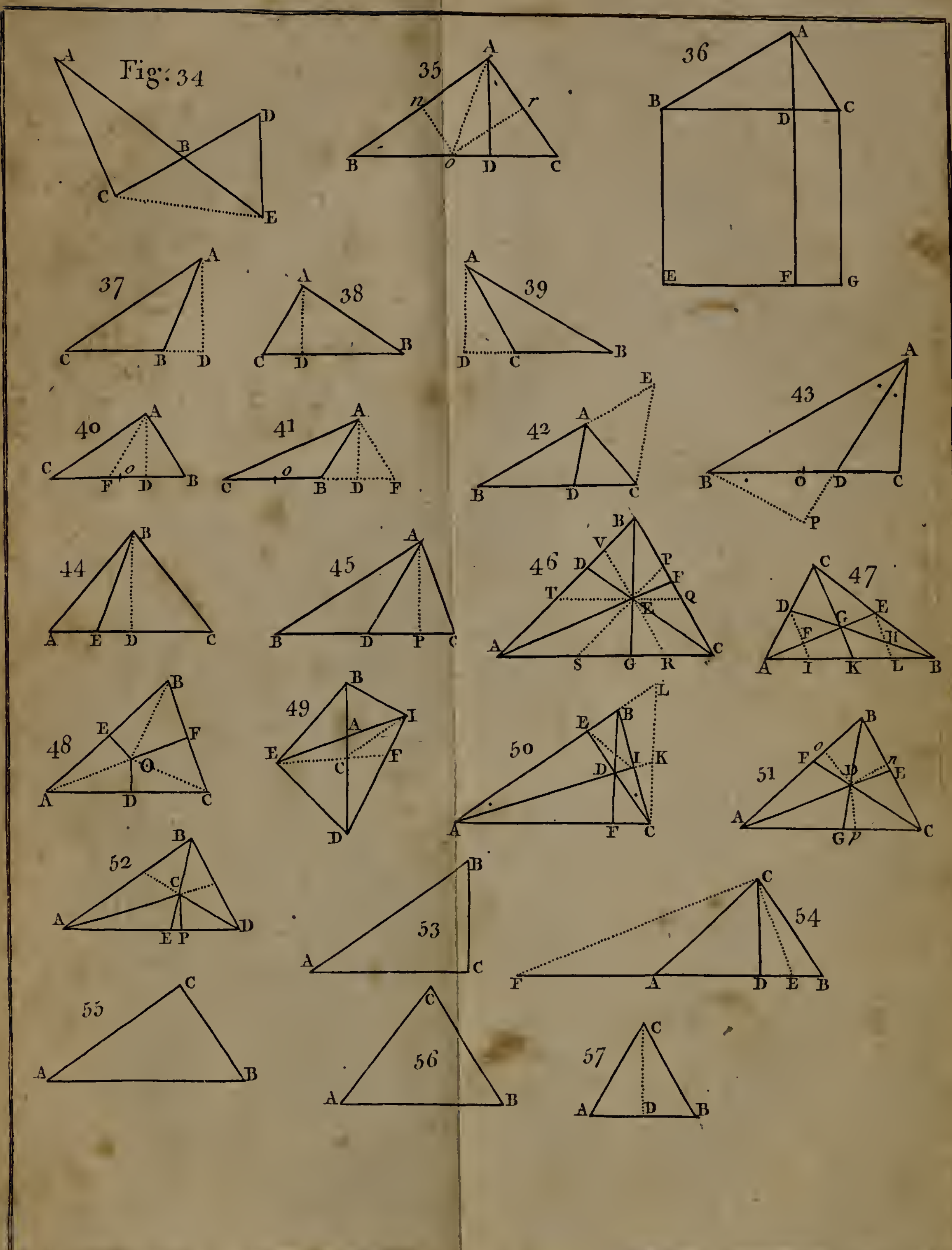
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Pl. II. *Pa.* 36.

10. The *diagonal* of a quadrangle or polygon, is a line drawn between any two oppoſite corners of the figure, as AB. FIG. 62.

11. The *hight* of a figure is a line drawn from the top, perpendicular to the *baſe*, or oppoſite ſide, on which it ſtands.

12. *Like or ſimilar figures*, are thoſe whoſe ſeveral angles are equal to one another; and the ſides about the equal angles, proportional.

13. *Homologous ſides* of two figures, are thoſe between two angles, reſpectively equal.

14. The *perimeter* or *circumference* of a figure, is the compaſs of it, or ſum of all the lines that incloſe it.

15. The *internal angles* of a figure, are thoſe on the inſide, made by thoſe lines that bound the figure, ADC. 76.

16. The *external angle* of a figure, is the angle made by one ſide of a figure, and the adjoining ſide drawn out, BAF. 77.

PROP. I.

In any parallelogram the oppoſite ſides, and angles, are equal; and the diagonal divides it into two equal triangles: AB = CD, AC = BD, *and triangle* ABD = ADC, *&c.* 63.

For ſince AB, and CD are parallel (Def. 2), ∠BAD = ADC (4. I): alſo, becauſe AC and BD are parallel, BDA = CAD (ibid.). Therefore the triangles ABD and DCA, are equal in all reſpects (7. II).

PROP. II.

The diagonals of a parallelogram, interſect each other in the middle.

In the triangles APC, BPD, ∠CAP = BDP, and ACP = DBP (4. I), and ∠BPD = APC (2. I), and AC = BD (Prop. I); therefore AP = PD, and CP = PB (7. II). 64.

FIG.

PROP. III.

Any line BC *passing through the middle of the diagonal of a parallelogram* P, *divides the area into two equal parts.*

65. For in the triangles ABP, and DCP, AP = PD (Prop. II); and all the angles are equal (4. I). Therefore the triangle ABP = DCP (7. II); and BP = PC (ibid.). And since triangle AED = AFD (Prop. I); the remainders BPDE and CPAF are equal; therefore BPDE + PDC = CPAF + APB, that is, EBCD = BAFC.

Cor. *Any right line* BC *drawn through the middle point* P *of the diagonal of a parallelogram, is bisected in that point*; BP = PC.

PROP. IV.

66. *In any parallelogram* ABDC, *the complements* CI, *and* IB, *are equal.*

For triangle ADC = ABD (Prop. I), and AHI = AGI, and IED = IFD (ibid.); therefore parallelogram HE = parallelogram GF (Ax. 4).

PROP. V.

66. *The parallelograms* HG, EF, *which are about the diameter* AD, *of any parallelogram* CB, *are similar to the whole* CB, *and to one another.*

The parallelograms HG, EF are equiangular to the whole CB (4. I), and to one another. The triangles ACD, ABD, as also AHI, AGI, and IED, IFD, are similar and equal (Prop. I). Therefore AH : HI or AG : : AC : CD or AB : : IE : ED or IF, therefore the parallelograms are like (Def. 12).

PROP.

PROP. VI.

FIG.

Parallelograms ABCD, *and* EBCF, *ſtanding upon the ſame baſe and between the ſame parallels, are equal.* 67.

For AD = BC = EF (Prop. I); add DE, then AE = DF, and AB = DC (Prop. I), and ∠A = CDF (Cor. 1. 4. I). Therefore triangle ABE ∓ DCF (6. II); ſubtract DGE; then the figure ABGD = EGCF; add BGC, then ABCD = BEFC.

Cor. 1. *Parallelograms of equal baſes and hights, are equal.*

For if their baſes be laid upon one another, the tops of both will fall in the ſame parallel, being of equal hight; and therefore they are equal (this prop.).

Cor. 2. *Every parallelogram is equal to the rectangle of its baſe and hight.*

Cor. 3. *Figures of the ſame area, may have their compaſs vaſtly different. And figures of equal compaſs may contain very different areas.*

PROP. VII.

A parallelogram is double to a triangle of the ſame or an equal baſe and hight.

For the triangle ACD = ABD (Prop. I), that is, the triangle ACD, on the baſe CD, is half the parallelogram ACDB on the ſame baſe CD, and between the ſame parallels. And ſince any triangle of an equal baſe and hight is = ACD, and any parallelogram of the ſame or an equal baſe and hight = ACDB. Therefore any triangle is half the parallelogram of the ſame or equal baſes and hights. 63.

 PROP.

FIG.

PROP. VIII.

68. *Parallelograms of the ſame hight, are to one another as their baſes*; DC : GF : : BC : GH.

Draw the diameters BA, EH. Then the triangles BCA, GHE, of the ſame hight, are as their baſes BC, GH (11. II). Therefore 2BCA : 2GHE : : BC : GH (Prop. V. Proportion): that is, parallelogram BCAD : parallelogram GHFE : : baſe BC : baſe GH.

Cor. 1, *Parallelograms of equal baſes, are as their hights.*

By Cor. 2. Prop. VI, as likewiſe

Cor. 2. *Parallelograms are to one another, as their baſes and hights.*

PROP. IX.

69. *Equal parallelograms having one angle equal to one; have the ſides about the equal angles reciprocally proportional. If* ABCD = EFGB, *then* AB : BG : : BE : BC.

Let the oppoſite angles at B be equal; produce DC and FG to H. Then AB : BG : : BD : BH (Prop. VIII) : : BF : BH (Ax. 6. Proportion) : : BE : BC (Prop. VIII).

Cor. 1. *Thoſe parallelograms are equal; which have one angle equal to one; and the ſides about the equal angles, reciprocally proportional.*

For BD : BH : : AB : BG (Prop. VIII) : : BE : BC (hyp.) : : BF : BH (Prop. VIII). Therefore parallelogram BD = parallelogram BF.

Cor. 2. *Equal parallelograms, have their baſes and hights, reciprocally proportional.*

Cor. 3. *If four lines are proportional; the rectangle of the means, is equal to the rectangle of the extremes.*

PROP.

PROP. X. FIG. 70.

Equiangular parallelograms AC, EG, *are in the complicate ratio of their homologous ſides,* ABC, EBG.

Produce DC, FG to H. Then parallelogram AC : BH : : AB : BG (Prop. VIII), and parallelogram BH : BF : : CB : BE (ibid.). Therefore parallelogram AC : parallelogram BF : : AB $\times$ CB : BG $\times$ BE (Cor. 1. Prop. XVIII. Proportion).

Cor. 1. *Parallelograms are to one another, in the complicate ratio of their baſes and hights.*

Cor. 2. *The rectangle of two lines, is a mean proportional between their ſquares.*

For ſuppoſing AC, EG, to be ſquares; then AC : BH : : (AB : BG : : BC : BE : :) BH : BF.

PROP. XI.

In any parallelogram AD, *the ſum of the ſquares of the diagonals, is equal to the ſum of the ſquares of all the ſides :* $AD^2 + CB^2 = CA^2 + AB^2 + BD^2 + DC^2$. 71.

For CE = EB, and AE = ED (Prop. II). Alſo $CD^2 + DB^2 = 2DE^2 + 2CE^2$ (28. II). And $2CD^2 + 2DB^2 = 4DE^2 + 4CE^2$, that is, $CD^2 + AB^2 + DB^2 + CA^2 = DA^2 + CB^2$.

PROP. XII.

If from any point O, *in the rectangle* AD, *lines be drawn to all the angles ; the ſum of the ſquares of the lines drawn to the oppoſite corners, will be equal:* $AO^2 + OD^2 = BO^2 + OC^2$. 72.

Draw AD, BC, to interſect in P, then AD = CB (6. II), and their halfs, AP = PC = PD. Then $CO^2 + OB^2 = 2CP^2 + 2OP^2$ (28. II) $= 2AP^2 + 2OP^2 = AO^2 + OD^2$ (28. II).

PROP.

PROP. XIII.

FIG. 73. *In any trapezium* ABDC, *let* E, F *be the middle points of the diagonals,* AD, BC. *Then the ſum of the ſquares of the ſides, is equal to the ſum of the ſquares of the diagonals, together with four times the ſquare of the diſtance, between the middle points of the diagonals:* $AB^2 + BD^2 + CD^2 + CA^2 = AD^2 + CB^2 + 4EF^2$.

For $AE^2 + ED^2 = 2AF^2 + 2EF^2$ (28. II). Alſo $AB^2 + AC^2 = 2CE^2 + 2AE^2$ (ibid.); alſo $BD^2 + DC^2 = 2CE^2 + 2DE^2$. And adding the two laſt equations, $AB^2 + BD^2 + DC^2 + CA^2 = 4CE^2 + 2AE^2 + 2ED^2 = CB^2 + 4AF^2 + 4EF^2 = CB^2 + AD^2 + 4EF^2$.

PROP. XIV.

74. *In any trapezium* ADBC, *let* E, F, *be the middle points of two oppoſite ſides. Then the ſum of the ſquares of the other two ſides, together with the ſquares of the diagonals, is equal to the ſum of the ſquares of the biſected ſides, together with four times the ſquare of the diſtance of theſe middle points:* $AC^2 + DB^2 + AB^2 + CD^2 = AD^2 + CB^2 + 4EF^2$.

Draw AE, ED. Then $AE^2 + ED^2 = 2AF^2 + 2EF^2$ (28. II), and $AB^2 + AC^2 = 2CE^2 + 2AE^2$ (ibid.), and $DB^2 + DC^2 = 2CE^2 + 2DE^2$ (ibid.): Add the two laſt equations, $AB^2 + AC^2 + DB^2 + DC^2 = 4CE^2 + 2AE^2 + 2ED^2 = CB^2 + 4AF^2 + 4EF^2 = CB^2 + AD^2 + 4EF^2$.

PROP. XV.

75. *In any trapezium* ADBC, *if lines be drawn to the middle of the oppoſite ſides; the ſum of the ſquares of the diagonals, is equal to twice the ſum of the ſquares of the biſecting lines:* $AB^2 + CD^2 = 2EF^2 + 2PQ^2$.

For

For $AB^2 + DC^2 + BD^2 + CA^2 = AD^2 + CB^2 + 4EF^2$ (Prop. XIV). FIG. 75.

And $AB^2 + DC^2 + BC^2 + DA^2 = AC^2 + DB^2 + 4PQ^2$ (ibid.).

and adding these equations,

$2AB^2 + 2DC^2 + BD^2 + CA^2 + BC^2 + DA^2 = AD^2 + CB^2 + AC^2 + DB^2 + 4EF^2 + 4PQ^2$, and subtracting what is common, $2AB^2 + 2DC^2 = 4EF^2 + 4PQ^2$, and $AB^2 + DC^2 = 2EF^2 + 2PQ^2$.

PROP. XVI.

The sum of the four internal angles of any quadrilateral figure, is equal to four right angles.

Draw the diagonal AC; then the sum of all the angles in the triangle ABC, or ADC, is two right angles (2. II); therefore the sum of both is four right angles. 76.

Cor. *If two angles of a quadrangle be right angles; the sum of the other two amounts to two right angles.*

PROP. XVII.

The sum of all the internal angles of a polygon, makes twice as many right angles, abating four, as the polygon has sides.

For drawing lines from all the angles, to a point O within the figure, it comes to be divided into as many triangles, as the figure has sides or angles. And each triangle contains two right angles (2. II), so these amount to twice as many right angles, as the figure has sides; but the angles at O are to be abated, and these amount to four right angles (Cor. 1. Prop. 1. I). 77.

Cor. *Hence all right-lined figures, of the same number of sides, have the sum of all the internal angles equal.*

PROP.

FIG.

PROP. XVIII.

The ſum of the external angles of any polygon, is equal to four right angles.

77. For all the internal angles, together with the external angles at the points A, B, C, *&c.* make twice as many right angles, as the figure has ſides (1. I); and the ſum of all the angles of the triangles ABO, BCO, *&c.* amounts to the ſame (2. II). Take away all the angles, EAB, ABC, *&c.* and there remains all the external angles A, B, C, *&c.* equal to all the angles at O, that is, four right angles (Cor. 1. (Prop. 1. I).

Cor. *All right-lined figures, have the ſum of their external angles equal.*

SCHOLIUM.

78. If any of the angles be greater than two right angles, as A; the external angle will run into the figure, and muſt be ſubtracted from the ſum of the reſt.

PROP. XIX.

79. *In two ſimilar figures* AC, PR; *if two lines* BE, QT, *be drawn after a like manner, as ſuppoſe, to make the angle* CBE = RQT; *then theſe lines have the ſame proportion, as any two homologous ſides of the figure,* BC *to* QR, *&c.*

Since ∠CBE = RQT, and R = C (hyp.); therefore BE : QT : : BC : QR (13. II) : : BA : QP (Def. 12) : : AD : PS (ibid.) : : DC : SR. Alſo BC : CE : : QR : RT; and BC : BE : : QR : QT, *&c.*

Cor. 1. *Hence all ſimilar figures are made up of ſimilar triangles.*

Draw

Draw BD, QS; and AC, PR; then BE : QT :: BC : QR (this prop.) :: CD : RS (Def. 12) :: CE : RT (this prop.) :: DE : ST (Prop. VIII. Proportion); therefore the triangles BCE and QRT are ſimilar; and BED and QTS are ſimilar. FIG. 79.

Again, the ∠ A = P, and AB : AD :: PQ : PS (Def. 12); therefore BAD, QPS are ſimilar (14. II). Alſo ∠ B = Q, and AB : BC :: PQ : QR, therefore ABC and PQR are ſimilar (14. II). Laſtly, ∠ D = S, and AD : DC :: PS : SR (Def. 12); therefore ADC, PSR are ſimilar (14. II).

Cor. 2. *Hence it may be laid down, as a diſtinguiſhing property of ſimilar figures, that they are made up of ſimilar triangles, placed in the ſame order.*

PROP. XX.

All ſimilar figures are to one another as the ſquares of their homologous ſides.

Let AD, PS be ſimilar polygons; draw AC, AD, PR, PS, which will divide the figures into triangles (Cor. 1. Prop. XIX). 80.

Becauſe AB : PQ :: AC : PR :: AD : PS (13. II); therefore

$AB^2 : PQ^2$ :: triangle ABC : PQR (18. II).

and $AB^2 : PQ^2 :: AC^2 : PR^2$ :: triangle ACD : PRS (ibid.).

and $AB^2 : PQ^2 :: AD^2 : PS^2$ :: triangle ADE : PST (ibid.).

therefore $AB^2 : PQ^2$ :: triangle ABC + ACD + ADE : triangle PQR + PRS + PST (Prop. X. Proportion) :: figure ABCDE : figure PQRST.

Cor. *If three lines* A, B, C *be in continual proportion; then as the firſt to the third, ſo any figure deſcribed on the firſt, to a ſimilar one upon the ſecond.*

For A : C :: $A^2 : B^2$ (Prop. XXIII. Proportion) :: figure upon A : figure upon B (this prop.).

PROP.

FIG.

PROP. XXI.

81. *If four lines be proportional,* AB : DE :: GH : LM ; *similar figures, alike described upon, two and two, shall also be proportional:* ABC : DEF :: GHIK : LMNO.

And if four figures be proportional, and two and two be similar ; their like sides shall be proportional.

For since AB : DE :: GH : LM (hyp.), therefore $AB^2 : DE^2 :: GH^2 : LM^2$ (Cor. 3. Prop. XVIII. Proportion).
whence ABC : DEF : GHI : LMN (Prop. XX).
Again, if the figures be similar,
and ABC : DEF :: GHIK : LMNO (hyp.).
then $AB^2 : DE^2 :: GH^2 : LM^2$ (Prop. XX).
whence AB : DE :: GH : LM (Cor. 3. 18. Proportion).

PROP. XXII.

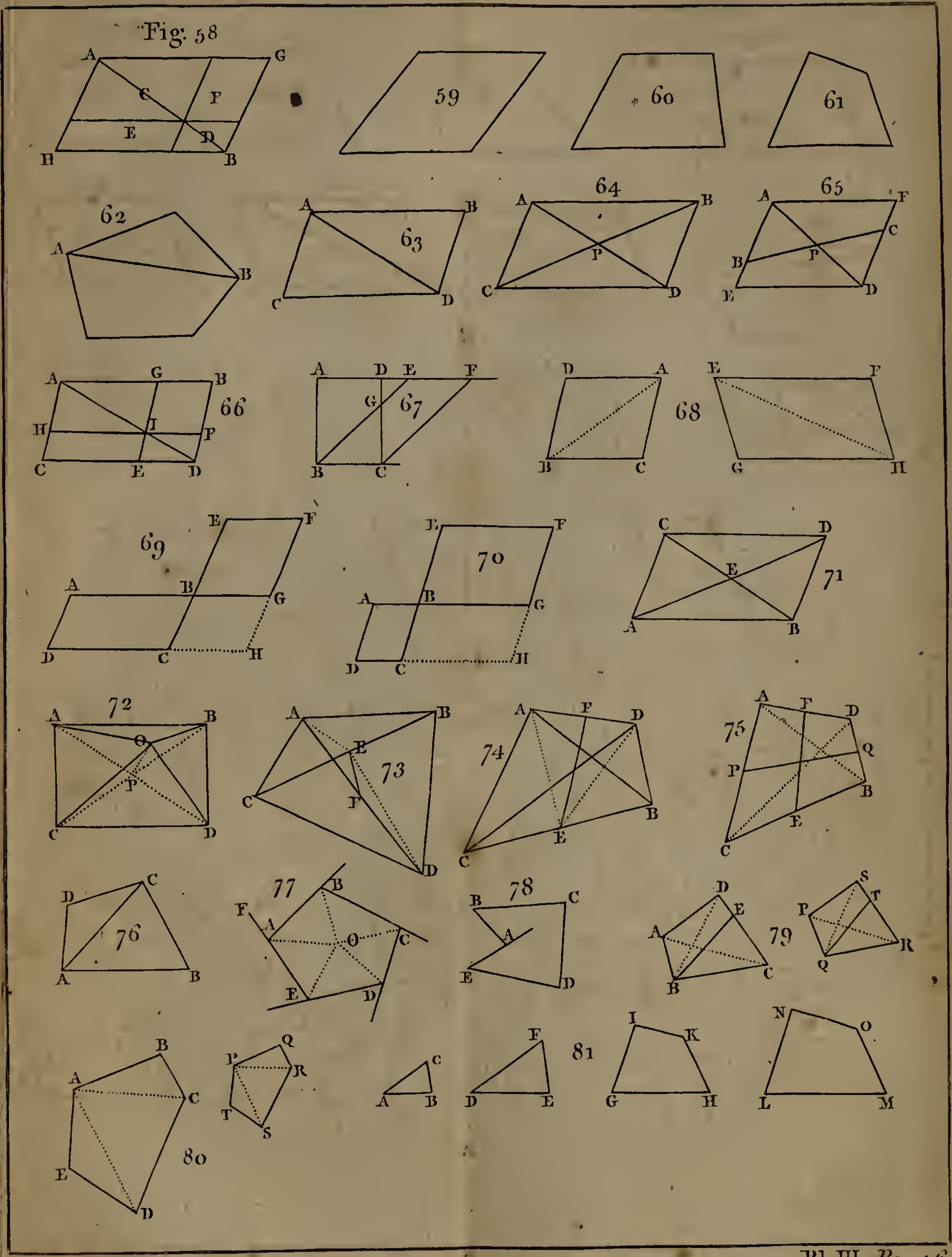
82. *Any figure described on the hypothenuse of a right-angled triangle, is equal to two similar figures described the same way upon the two sides:* BFC = ALC + AGB.

For fig. BCF : $\begin{matrix}\text{CAL}\\\text{BAG}\end{matrix}$ :: BC^2 : $\begin{matrix}CA^2\\AB^2\end{matrix}$ (Prop. XX).
therefore, BCF : CAL + BAG :: $BC^2 : CA^2 + AB^2$ (14. Proportion).
But $BC^2 = CA^2 + AB^2$ (21. II) ; therefore BCF = CAL + BAG (Prop. II. Proportion).

PROP. XXIII.

83. *The area of a trapezoid* ABCD, *is equal to the rectangle of half the sum of the parallel sides, and the perpendicular between them:* $\frac{BA + CD}{2} \times BP$.

Draw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81

Draw AF parallel to BD, and BP perpendicular to CD. Then the area ABDF = AB × BP or FD × BP (Cor. 2. Prop. VI) = $\frac{AB + FD}{2}$ × BP. And the area of the triangle ACF = $\frac{1}{2}$CF × BP (Cor. 2. (Prop. X. II). Therefore CAF + AFDB or the trapezoid CABD = $\frac{AB + DF + CF}{2}$ × BP = $\frac{AB + CD}{2}$ × BP. FIG. 83.

PROP. XXIV.

The area of a trapezium ABDF, *is equal to half the rectangle under the diagonal* AD, *and the sum of the perpendiculars falling thereon from the opposite angles :* AD × $\frac{BC + EF}{2}$. 84.

For the triangle ABD = $\frac{AD \times BC}{2}$ (Cor. 2. 10. II); and the triangle AFD = $\frac{AD \times FE}{2}$ (ibid.): therefore ABD + AFD or the trapezium ABDF = $\frac{AD \times \overline{BC + FE}}{2}$.

PROP. XXV.

Any regular figure ABCDE, *is equal to a triangle, whose base is the perimeter* ABCDEA ; *and hight, the perpendicular* OP, *drawn from the center, perpendicular to one side.* 85.

Two perpendiculars, as PO, standing on the middle of two sides, meet in the center, O (9. II). Or two angles A, B bisected by two right lines, meet in the center, O (Cor. 1. 3. II): whence all the lines OA, OB, OC are equal ; and all perpendiculars drawn from O, upon AB, BC, CD, *&c.* are equal. And all the triangles AOB, BOC, *&c.* are

3 equal

FIG. 85. equal and fimilar. The fum of all the triangles make up the figure, that is, $\frac{AB \times OP}{2}$, $\frac{BC \times OP}{2}$, &c. or $\frac{AB + BC + CD + DE + EA}{2} \times OP =$ figure, or a triangle whofe bafe is ABCDEA, and hight OP $=$ the figure.

Cor. *The area of a regular polygon is equal to the rectangle of one fide into the perpendicular from the center upon that fide, and that multiplied by half the number of fides.*

SCHOLIUM.

Any polygon, regular or irregular, may be divided in as many triangles, lefs 2, as the figure has fides; by drawing diagonal lines.

PROP. XXVI.

Only three forts of regular figures can fill up a plane furface; and thefe are fix triangles, four fquares, and three hexagons.

It is required to place fome number of thefe figures, with their angles upon one point, fo that being joined clofe together, they may fill the whole fpace around it, and leave no vacancy.

It is plain the angles about one point are four right angles (Cor. 1. 1. I), which want to be filled up. Now if the angles of the feveral figures be computed by Prop. XVII, they will be found as follows.

86. A triangle $\frac{2}{3}$ of a right angle $=$ A.
A fquare 1 right angle $=$ B.
A pentagon $1\frac{1}{5}$ right angle.
A hexagon $1\frac{1}{3}$ a right angle $=$ C.
&c.

Now $\frac{2}{3}$ of a right angle 6 times repeated, makes 4 right angles, and therefore fills all the fpace; that is, 6 angles of an equilateral triangle fills it.

Alfo

Alſo 4 angles of a ſquare (or 4×1), makes 4 FIG.
right angles. 86.

But 3 angles of a pentagon (or $3 \times 1\frac{1}{5}$) falls ſhort; and 4 angles (or $4 \times 1\frac{1}{5}$) exceeds.

Alſo 3 angles of a hexagon (or $3 \times 1\frac{1}{3}$) makes 4 right angles. And theſe are all; for

The angle of a heptagon (and other figures) is bigger, and therefore 3 angles will exceed 4 right ones. And to have two angles, each muſt be right angles, which is abſurd.

BOOK IV.

Of the Circle, and inscribed and circumscribed Figures.

DEFINITIONS.

FIG. 87. 1. A *Circle* is a plane figure described by a right line moving about a fixt point, ABD. Or it is a figure bounded by one line equidistant from a fixt point.

2. The *center* of a circle, is the fixt point about which the line moves, C.

3. The *radius*, is the line that describes the circle, CA.

Cor. *All the radii of a circle, are equal.*

4. The *circumference* is the line described by the extreme end of the moving line, ABDA.

88. 5. The *diameter*, is a line drawn through the center, from one side to the other, AD.

6. A *semicircle*, is half the circle, cut off by the diameter, as ABD.

7. A *quadrant*, or quarter, is the part between two radii perpendicular to one another, as CDE.

89. 8. An *arch* is any part of the circumference, AB.

9. A *sector*, is a part bounded by two radii, and the arch between them, ACB.

10. A *segment*, a part cut off by a right line, DEF, or DABF.

11. A

11. A *cord*, a right line drawn through the circle, as DF. FIG. 89.

12. *Angle at the center*, is that whoſe angular point is at the center ACB.

13. *Angle at the circumference*, is when the angular point is in the circumference, BAD. 90.

14. *Angle in a ſegment*, is the angle made by two lines drawn from ſome point of the arch of that ſegment, to the ends of the baſe; as BCD is an angle in the ſegment BCD.

15. *Angle upon a ſegment*, is the angle made in the oppoſite ſegment, whoſe ſides ſtand upon the baſe of the firſt; as BAD, which ſtands upon the ſegment BCD.

16. A *tangent* is a line touching a circle, which produced, does not cut it, as GAF.

17. Circles are ſaid *to touch* one another, which meet, but do not cut one another.

18. *Similar arches*, or *ſimilar ſectors*, are thoſe bounded by radii that make the ſame angle.

19. *Similar ſegments* are thoſe which contain ſimilar triangles, alike placed.

20. A *figure* is ſaid to be *inſcribed in a circle*, or a *circle circumſcribed about a figure*; when all the angular points of the figure are in the circumference of the circle.

21. A *circle* is ſaid to be *inſcribed* in a figure, or a *figure circumſcribed about* a circle; when the circle touches all the ſides of the figure.

22. One *figure* is *inſcribed* in another, when all the angles of the inſcribed figure, are in the ſides of the other.

PROP. I.

The cord of any arch AB, *falls intirely within the circle.* 91.

For draw CA, CB; and CD to any point of the cord; then ∠A = B (3. II). And ∠CDB =

FIG. 91. A + ACD (1. II) = B + ACD, therefore CDB is greater than B, consequently CB is greater than CD, (4. II); therefore D is within the circle.

PROP. II.

92. *The radius* CR, *bisects any cord at right angles, which passes not through the center, as* AB.

For draw AC, BC, and if AF = FB, then since AC = CB, and CF common; therefore CFA = CFB (8. II) = a right angle; and angle ACF = BCF.

Or if AFC = CFB and A = B, then ACF = BCF; and CF being common, AF = FB. This prop. follows from Cor. 3. Prop. III. Book II.

Cor. 1. *If a line bisects a cord at right angles, it passes through the center of the circle.*

Cor. 2. *The radius that bisects the cord, also bisects the arch.*

For since ACR = RCB. If CBR be laid upon CAR, the point B will fall upon A, and therefore RB = RA.

Cor. 3. *If two right lines do not both pass through the center, they cannot both be bisected by each other.*

For if they could, they must both make right angles with the radius.

PROP. III.

93. *In a circle, equal cords* AB, GD, *are equally distant from the center,* C.

For let CE, CF be perp. to the cords; and draw CD, CA; then in the triangles, ACF, DCE, AC = CD, AF = DE being half the cords (Prop. II); and angles at F, E right; and the angles at C, both acute, therefore CF = CE (9. II).

Cor.

Cor. *If several lines be drawn through a circle, the greatest is the diameter, and those that are nearer the center* HI, *are greater than those that are farther off,* DG. FIG. 93.

For draw CH, then CH is greater than OH (4. II), and therefore 2CH or the diameter is greater than HI. And ſince ∠HCI is greater than DCG, HI is greater than DG (Cor. 6. II).

PROP. IV.

If from a point G, *out of the center, ſeveral lines* GD, GE, *&c. be drawn, the greateſt is that* GF *which paſſes through the center, and thoſe nearer to* GF *are greater than thoſe further off.* 94.

Alſo GH *(the remainder to* GF*) is the leaſt, and thoſe nearer to it, as* GA, *are leſs than thoſe further off,* GB.

Draw CE, CD, CA, CB, from the center C. Then GC + CE or GF is greater than GE (5. II). Alſo in the triangles GCE, GCD; GC, EC are equal to GC, DC; but ∠ECG is greater than DCG; therefore EG is greater than DG.

Alſo CG + GD is greater than CD or CH, take away CG, and GD is greater than GH. After the ſame manner GA is greater than GH; and GB greater than GA.

Cor. 1. *Only two lines drawn from* G *to the circumference can be equal; and lie on different ſides of the diameter* HF.

For no two lines on the ſame ſide can be equal.

Cor. 2. *If from any point, three equal right lines can be drawn to the circumference; that point is the center,* C.

Cor. 3. *No circle can cut another in more than two points.*

FIG. For then three equal lines might be drawn from
94. a point out of the center to the circumference; which is absurd.

PROP. V.

95. *If from a point* G *without a circle, several right lines be drawn to cut it. Of those that pass to the concave part, the greatest is that* GF *which passes through the center, and those nearer to* GF *are greater than those further off.*

But of those that go to the convex part, the least is that GH, *which continued would pass through the center, and those nearer to that, as* GA, *are less than those further off,* GD.

For in the triangle GCE, GC + CE or GF is greater than GE. And in the triangles GCE, GCB; GC, CE are equal to GC, CB, and ∠GCE greater than GCB, therefore GE is greater than GB.

Also in the triangle CGA, CA + AG is greater than CG or CH + HG (5. II); take away CA = CH, and AG is greater than HG. And in the triangles CAG, CDG; CG, CA are equal to CG, CD; and angle GCA less than GCD; therefore GA is less than GD (Cor. 6. II).

Cor. 1. *There can only two equal lines be drawn from the point* G *to the circumference of the circle.*

For no two are equal on one side of GF.

Cor. 2. *The greatest to the convex part, or the least to the concave part, is the tangent to the circle.*

PROP. VI.

96. *In any circle, if several radii be drawn making equal angles; the arches and sectors comprehended thereby will be equal, if* ACB = BCD; *then, arch* AB = *arch* BD; *and sector* ACB = BCD.

For

For ſince ∠ACB = BCD, and CA = CD; FIG. therefore if the angle DCB be laid upon BCA, DC 96. will fall upon CA, and D upon A, and conſequently the arch DB will coincide with AB, as well as the ſector DBC with ABC, conſequently arch DB = AB, and ſector DBC = ABC (Ax. 8).

Cor. 1. *In equal circles, the radii making equal angles, comprehend equal arches, and ſectors.*

Cor. 2. *In the ſame or equal circles, the radii making equal angles, comprehend equal cords* AB, BD.

For theſe will coincide with one another. It alſo follows from Prop. VI. II.

Cor. 3. *Equal cords cut off equal arches, and equal ſegments, in the ſame circle.*

For if laid upon one another, they perfectly coincide, as has been proved.

PROP. VII.

In the ſame or equal circles, the arches, and alſo the 97. *ſectors, are proportional to the angles intercepted by the radii.*

Take any arch AB as ſmall as you will, and let AB = BD, &c. alſo AB = QR = RS, &c. and drawing CA, CB, CD, &c. and PQ, PR, PS, &c. then all the angles ACB, BCD, QPR, RPS, &c. are equal (Cor. 1. Prop. VI). Whence AF is as multiple of AB, as the angle ACF is of ACB. Therefore AB : AF :: ACB : ACF (Prop. V. Proportion). Alſo QV is as multiple of QR or AB, as QPV is of QPR or ACB, whence AB : QV :: ACB : QPV (ibid.); whence AF : QV :: ACF : QPV (Cor. 2. 14. Proportion).

The ſame reaſoning holds in the ſectors, for ſect. ACF is as multiple of ACB; as ∠ACF is of the ∠ACB. And ſect. QPV is as multiple of QPR or ABC;

FIG. ABC; as ∠ QPV is of ACB. Therefore sect.
97. ACF : sect. QPV : : angle ACF : ∠ QPV.

Cor. *The angle* ACF *is to* 4 *right angles; as the arch* AF, *is to the whole circumference.*

PROP. VIII.

98. *In all circles, similar arches are as the radii of the circles.*

Let the circles AFG and *afg* be both described from the same center, C. Draw the radii CA, CF; then the arches AF, *af* are similar (Def. 18). Draw CB extremely near CA. Then the figures or sectors C*ab*, CAB, approach very near to isosceles triangles, which are similar to one another, because the ∠ at C is common (3. II). Therefore C*a* : *ab* : : CA : AB (13. II); and C*a* : CA : : *ab* : AB (4. Proportion). Now if you suppose BF divided into more arches, equal to AB; and more radii CB drawn; *bf* will then contain as many arches equal to *ab*. Therefore *af* is as multiple of *ab*, as AF is of AB; therefore *ab* : AB : : *af* : AF (5. Proportion); whence C*a* : CA : : *af* : AF (1. Proportion).

PROP. IX.

98. *The circumferences of circles are to one another, as their diameters.*

For AF : circumference AFGA : ∠ ACF : 4 right angles (Cor. 7) : : ∠ *a*C*f* : 4 right angles : : *af* : circumference *afga*. And AFGA : *afga* : : AF : *af* (4. Proportion) : : CA : *ca* (Prop. VIII) : : 2CA : 2 C*a* (5. Proportion).

Cor. *The circumferences of circles are as their radii.*

PROP.

PROP. X.

FIG. 88.

A right line AG, *perpendicular to the diameter* AD *of a circle, at the extreme point* A, *touches the circle in that point; and lies wholly without the circle.*

To any point O in the line GAF, draw the line CO from the center. Then the hypothenuſe OC is greater than the ſide AC (4. II). Therefore O is without the circle. And ſo it is for any point beſides A; therefore the line GF is entirely out of the circle.

Cor. 1. *Hence a right line touches a circle only in one point.*

Cor. 2. *If a right line touches a circle in one point, it is perpendicular to the diameter in that point.*

Cor. 3. *All circles, whoſe centers are in the line* AD, *and whoſe circumferences paſs through the point* A, *touch one another, and the line* GAF, *in the ſame point* A.

Cor. 4. *Hence, if two circles touch one another, either inwardly or outwardly; the line paſſing through their centers,* C, B, D, *ſhall alſo paſs through the point of contact,* A. 99.

Otherwiſe a line, touching both circles in that point, could not be perpendicular to both diameters.

Cor. 5. *Two circles, can only touch in one point.*

From the centers B, D, draw BO, DO, to a point O in the exterior circle. Then in the triangle BOD; DB + BO is greater than DO or DA or DB + BA (5. II). Whence BO is greater than BA; therefore the point O, is without the circle AE. In like manner, drawing CO; DO + CO is greater than DA + CA, and CO greater than CA, therefore O falls without the circle AI.

PROP.

FIG. **PROP. XI.**

100. *The angle of contact between a right line and a circle* DAI, *is less than any right-lined angle whatever,* DAL.

Draw BE perpendicular to AL, then the side BA opposite to the right angle BEA, is greater than the side BE opposite to the acute angle BAE (4. II). Therefore the point E, and so the whole line AEL, falls within the circle.

Cor. 1. *Hence the angle of a semicircle* BAI *is greater than any acute angle whatever.*

Cor. 2. *The angle of contact* DAI, *is infinitely less than a right angle.*

For if it was in a finite proportion to a right angle, then an acute angle might be found equal to it.

Cor. 3. *If any other circle be described through* A, *with any radius greater than* AB, *it will fall entirely between the tangent* AD *and the circle* AL, *and make the angle of contact less. And circles may be described* ad infinitum, *which shall only touch one another in* A; *their centers being all in the line* AB *produced.*

All this appears by Cor. 5. Prop. X. compared with this prop.

PROP. XII.

101. *In a circle, the angle at the center is double the*
102. *angle at the circumference, standing upon the same arch*; BDC = 2BAC.

Case 1. When one side AF passes through the center; in the isosceles triangle ADC, ∠DAC = DCA (3. II), and the ∠FDC = DAC + DCA (1. II) = 2FAC.

Case 2. If the center of the circle be within the angle BAC; draw ADF, then by Case 1, FDC = 2FAC,

2FAC, and FDB = 2FAB, therefore the whole BDC = 2BAC. FIG. 101.

Case 3. If the center of the circle be without the angle, BAC; draw ADF, then by Case 1, FDB = 2FAB, and FDC = 2FAC, therefore the remainder BDC = 2BAC (Ax. 4). 102.

Cor. 1. *The angle at the circumference standing upon any arch, is equal to half the angle at the center, upon the same arch; or to the angle at the center upon half the arch.*

Cor. 2. *In the same or equal circles, the angles at the circumference, are equal, which stand upon equal arches or equal cords.*

This is plain from Cor. 1, 2. Prop. VI.

PROP. XIII.

All angles in the same segment of a circle, are equal, DAC = DBC, *and* DGC = DHC. 103.

For ∠DGC and DHC are each equal to the angle at the center, on half the arch DABC. And DAC, DBC are each of them equal to the angle at the center, on half the arch AGHC.

Or thus.

The ∠DGC = $\frac{1}{2}$DOC = DHC (Prop. XII). Again, ∠DFC = DAF + ADF (1. II) = DBC + BCF (ibid.), but ADF = BCF (Prop. XII); therefore DAF or DAC = DBC (Ax. 4).

Cor. *If the extremities of two equal arches* DA, BC, *be joined by right lines,* DC, AB; *they will be parallel.*

For ∠BAC = DCA (Cor. 2. 12), therefore AB, CD are parallel (Cor. 3. 4. I).

PROP.

PROP. XIV.

FIG. 104. *The angle* ABC *in a ſemicircle is a right angle.*

For draw BD to the center, then BDA, BDC are two iſoſcèles triangles, therefore DAB = DBA, and DCB = DBC (3. II). And DAB + DCB = DBA + DBC = ABC (Ax. 3) = half of two right angles (2. II) = a right angle.

Cor. 1. *The angle* ABG, *in a greater ſegment* ABFG, *is leſs than a right angle; and the angle* ABF, *in a leſs ſegment* ABF, *is greater than a right angle.*

This is evident by inſpecting the figure.

Cor. 2. *If a line be drawn from the middle of the hypothenuſe (of a right-angled triangle), to the right angle; it cuts the triangle into two iſoſceles triangles.*

PROP. XV.

105. *If two lines cutting a circle, interſect one another in* A; *and there be made at the center,* ∠ECF = BAD;

Then arch BD + GH = 2EF, *if* A *is within the circle; or arch* BD — GH = 2EF, *if* A *is without.*

For draw HI parallel to GD, then DI = GH (Cor. 13); and angle BHI = BAD = ECF (4. I). Therefore EF = $\frac{1}{2}$BI (Cor. 1. 12); and 2EF = BI = BD + GH, when A is within, but = BD — GH, when A is without the circle.

Cor. 1. *If from a point without, two lines touch a circle; the angle made by them is equal to the angle at the center, ſtanding on half the difference, of theſe two parts of the circumference.*

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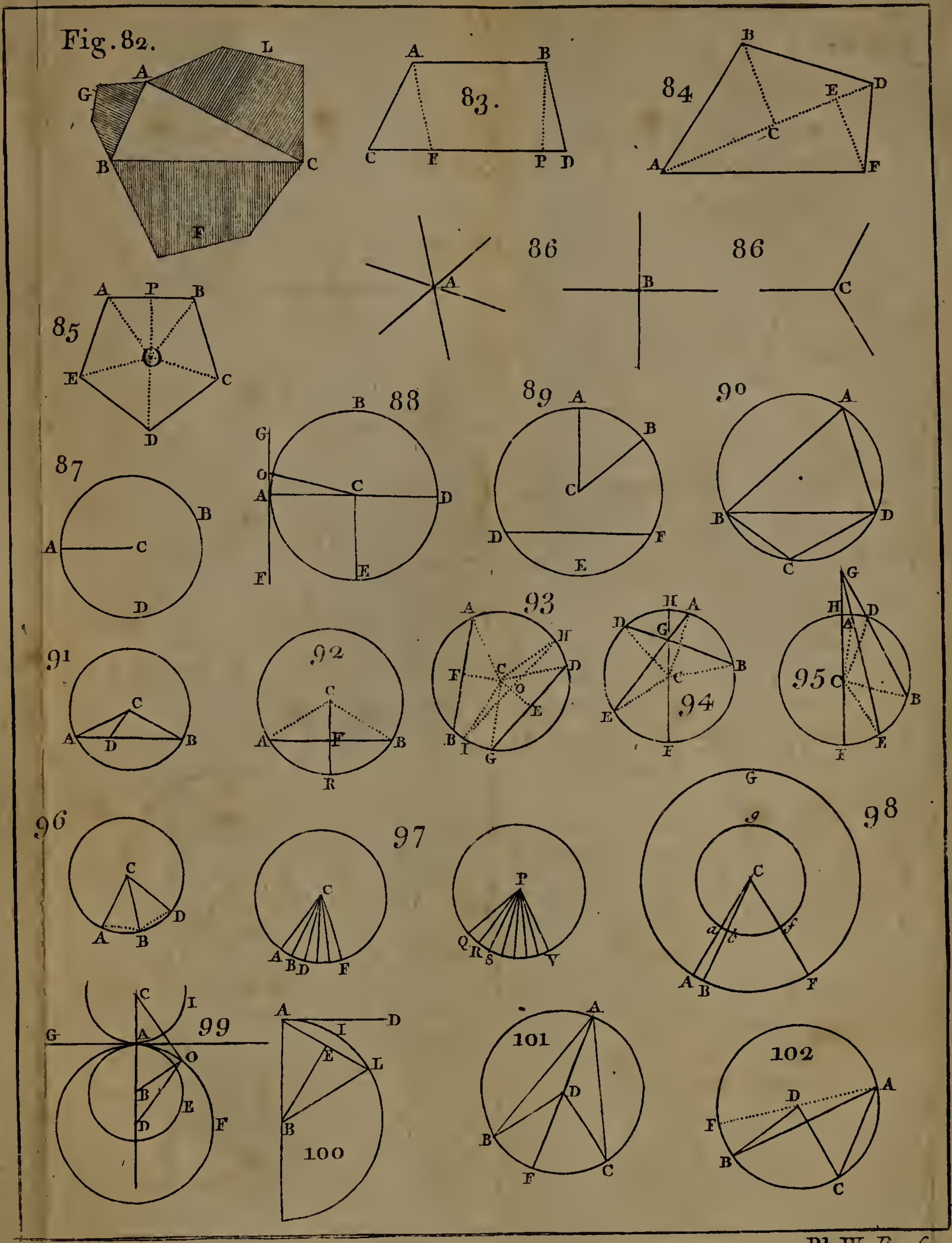
Fig. 82.
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Pl. IV. Pa 65.

This is plain, by supposing B, H, and G, D to coincide in the periphery, then half their difference will be = EF. FIG. 105.

Cor. 2. *The angle* A = ∠BHD + HDG, *when* A *is within*; *or* A = BHD − HDG, *when* A *is without the circle* (1. II).

PROP. XVI.

In a circle, the angle made at the point of contact between the tangent and any cord, is equal to the angle in the alternate segment; ECF = EBC, *and* ECA = EGC. 106.

Through the center O, draw the diameter COD, which is ⊥ to CF (Cor. 2. 10). The ∠CED is right (Prop. XIV); therefore ∠D + DCE = a right angle (Cor. 2. 2. II) = DCE + ECF; therefore D = ECF, or EBC = ECF (Prop. XIII).

Again, CEG + ECG + G = two right angles (2. II) = GCF + ECG + ECA (1. I), CEG + G = GCF + ECA, but CEG = GCF (this prop.), therefore G = ECA (Ax. 4).

Cor. *A tangent to the middle point of an arch, is parallel to the cord of it.*

For if arch CB = CE, then cord CB = cord CE (Prop. VI. and Cor. 2); whence ∠E = B = ECF (this prop.), whence BE, CF are parallel (Cor. 3. 4. I).

PROP. XVII.

If from any point B *in a semicircle, a perpendicular* BD *be let fall upon the diameter, it will be a mean proportional between the segments of the diameter*: AD : DB : : DB : DC. 107.

For drawing AB, BC, the triangles ABC, ABD, DBC are similar, for ∠ABC is right (Prop. XIV),

 and

FIG. 107. and angles at D are right, and BAD = BAC, ABD = BCD, and therefore DBC = BAD. Therefore AD : DB : : DB : DC (13. II).

Cor. *The cord is a mean proportional between the adjoining ſegment, and the diameter*; AD : AB : AC. *And* CD : CB : CA ∻.

This is evident from the ſimilarity of the triangles.

PROP. XVIII.

108. *In a circle, if the diameter* AD *be drawn, and from the ends of the cords* AB, AC, *perpendiculars be drawn upon the diameter; the ſquares of the cords will be as the ſegments of the diameter*; AE : AF : : AB^2 : AC^2.

For AE × AD = AB^2 (Cor. 17), and AF × AD = AC^2 (ibid.); therefore AB^2 : AC^2 : : AE × AD : : AF × AD : : AE : AF (Cor. 1. 5. Proportion).

PROP. XIX.

109. *If two circles touch one another in* P, *and the line* PDE *be drawn through their centers; and any line* PAB *is drawn through that point, to cut the circles, that line will be divided in proportion to the diameters*; PA : PB : : PD : PE.

For drawing AD, BE; the triangles PAD, PBE, are right-angled at A, B (14), and conſequently ſimilar, therefore PD : PE : : PA : PB (13. II).

Cor. *The arches* AD, BE *are ſimilar; as alſo the arches* PA, PB; *and theſe arches are as the whole circumferences of the circles, or as the diameters*; AD : BE : : PA : PB : : PD : PE, *&c.*

They are ſimilar by Def. 18. and proportional by Prop. VIII.

PROP.

PROP. XX. FIG.

If through any point F *in the diameter of a circle, any cord* CFD *is drawn; the rectangle of the segments of the cord, is equal to the rectangle of the segments of the diameter;* CFD = AFB. 110.

Draw AC, BD; then the triangles CAF, BDF are similar, for the angle F is common, and CAF = BDF, and ACF = DBF (Cor. 2. 12); therefore AF : FC :: FD : FB (13. II), and AF × FB = CF × FD (12. Proportion).

Cor. 1. *Let* O *be the center; then the rectangle* CFD, *is equal to radius square — the square of the distance from the center;* CFD = $AO^2 - OF^2$.

For AF × FB = $\overline{AO + OF} \times \overline{AO - OF} = AO^2 - OF^2$ (12. I).

Cor. 2. *If several cords* CD, EG, *be drawn through the same point* F, *the rectangles of their segments will be all equal to one another;* CFD = EFG.

For they are all equal to the rectangle AFB.

PROP. XXI.

If through any point F *out of the circle in the diameter* BA *produced, any line* FCD *is drawn through the circle; the rectangle of the whole line and the external part, is equal to the rectangle of the whole line passing through the center, and the external part;* DFC = AFB. 111.

For drawing DA, CB, the triangles DFA and BFC are similar; for ∠FDA = FBC, and F is common; therefore AF : FD :: CF : FB (13. II); and AF × FB = CF × FD.

Cor.

FIG. Cor. 1. *Let* O *be the center, then the rectangle*
111. CFD *is equal to the square of the distance from the center — radius square*; CFD $=$ FO^2 — AO^2.

For AF $\times$ FB $= \overline{FO - AO} \times \overline{FO + AO} =$ FO^2 — AO^2 (12. I).

Cor. 2. *Let* HF *be a tangent at* H; *then the rectangle* CFD $=$ *square of the tangent* FH.

For FO^2 — AO^2 $=$ FO^2 — OH^2 $=$ FH^2 (Cor. 1. 21. II).

Cor. 3. *If several lines* FD, FG, *are drawn from the same point* F; *the rectangles of the whole and external segment, will be all equal to one another*; CFD $=$ EFG.

For they are all equal to the rectangle AFB.

Cor. 4. *If from the same point* F, *two tangents be drawn to the circle, they will be equal*; FH $=$ FI.

For the square of either of them is equal to the rectangle AFB.

PROP. XXII.

112. *If a line* PFC *be drawn perpendicular to the diameter* AD *of a circle*; *and any line drawn from* A *to cut the circle and perpendicular*; *then the rectangle of the distances of the sections from* A, *will be equal to the rectangle of the diameter and the distance of the perpendicular from* A; AB $\times$ AC $=$ AP $\times$ AD.

For draw BD, and the triangles ABD, APC are similar, for $\angle$ at A is common, and $\angle$P and B are right (14); therefore AD : AB :: AC : AP (13. II), and AD $\times$ AP $=$ AB $\times$ AC (12. Proportion).

Cor. 1. *If* PF *cuts the circle in* K, *then* AB $\times$ AC $=$ AK^2.

Cor.

Cor. 2. *If more lines* AEF *be drawn, all the rectangles* EAF, BAC *are equal.* FIG. 112.

For they are all equal to the rectangle PAD.

PROP. XXIII.

In a circle EDF *whose center is* C, *and radius* CE, *if the points* B, A, *be so placed in the diameter produced, that* CB, CE, CA *be in continual proportion, then two lines* BD, AD *drawn from these points, to any point in the circumference of the circle, will always be in the given ratio of* BE *to* AE. 113.

Draw DP perpendicular to the diameter EF, then $DP^2 = EP \times PF$ (17) $= 2CE \times EP - EP^2$, whence $AD^2 = \overline{AE + EP}^2 + PD^2$ (21. II) $= AE^2 + EP^2 + 2AEP + 2CEP - EP^2$ (10. I) $= AE^2 + 2CE \times EP + 2AE \times EP$. Also $BD^2 = \overline{BE - EP}^2 + PD^2$ (21. II) $= BE^2 - 2BEP + EP^2 + 2CEP - EP^2$ (11. I) $= BE^2 + 2CE \times EP - 2BE \times EP$.

And since CA : CE : CB $\div\!\div$, therefore AE : CE : : EB : CB (13. Proportion), or AE : EB : : CE : CB (4. Proportion). Also $AE^2 : EB^2 : : CE^2 : CB^2 : :$ CA : CB (23. Proportion) : : CE + AE : CE − EB : : $2CE \times EP + 2AE \times EP : 2CE \times EP - 2EB \times EP$ (5. Proportion). And $AE^2 : EB^2 : : AE^2 + 2CE \times EP + 2AE \times EP : EB^2 + 2CE \times EP - 2EB \times EP$ (10. Proportion) $: : AD^2 : BD^2$. And AE : EB : : AD : BD (Cor. 3. 18. Proportion).

PROP. XXIV.

If D, C *be two points in the diameter of a circle, equidistant from the center* O; *and if two lines be drawn from thence to any point* E, *in the circumference, the sum of their squares will be equal to the sum of the squares of the segments of the diameter*; $DE^2 + CE^2 = AC^2 + CB^2$. 114.

For

FIG. 114. For draw EO to the center O, then $DE^2 + CE^2 = 2DO^2 + 2OE^2$ (28. II) $= 2AO^2 + 2OC^2$. But $AC^2 + CB^2 = \overline{AO + OC}^2 + \overline{AO - OC}^2 = AO^2 + OC^2 + 2AOC + AO^2 + OC^2 - 2AOC$ (10. I) $= 2AO^2 + 2OC^2 = DE^2 + CE^2$.

Cor. 1. *Hence the ſum of the ſquares of* DE, CE *is equal to twice the ſquare of the radius* + *twice the ſquare of the diſtance of one of the points from the center*; $DE^2 + CE^2 = 2AO^2 + 2OC^2$.

Cor. 2. *The ſum of the ſquares of any two correſpondent ones will be equal.*

For they are all equal to the ſame given quantity.

PROP. XXV.

115. *If any cord* PQ *be drawn parallel to the diameter* AB, *of a circle; and from a given point* C *in that diameter, the lines* CP, CQ *be drawn to the two ends of the cord; I ſay the ſum of their ſquares is equal to the ſum of the ſquares of the ſegments of the diameter*; $CP^2 + CQ^2 = AC^2 + CB^2$.

For draw PS, QR $\perp$ to the diameter AB, then PS^2 or $QR^2 = PC^2 - SC^2 = QC^2 - RC^2$ (21. II); that is, $PC^2 - \overline{SO + OC}^2 = QC^2 - \overline{SO - OC}^2$; or $PC^2 - SO^2 - 2SOC - OC^2$ (10. I) $= QC^2 - SO^2 + 2SOC - OC^2$, becauſe $OR = OS$. Therefore $PC^2 = QC^2 + 4SOC$, but $AC^2 + CB^2 = \overline{AO + OC}^2 + \overline{AO - OC}^2 = 2AO^2 \times 2OC^2$ (10, 11. I). But $PC^2 = AO^2 + OC^2 + 2SOC$ (22. II) $= QC^2 + 4SOC$. Therefore

$$QC^2 = AO^2 + OC^2 - 2SOC$$
$$PC^2 = AO^2 + OC^2 + 2SOC$$

therefore $PC2 + QC^2 = 2AO^2 + 2OC^2 = AC^2 + CB^2$.

Cor.

Cor. 1. *The sum of their squares*, $PC^2 + QC^2 = 2AO^2 + 2OC^2$. FIG. 115.

Cor. 2. *The difference of their squares*, $PC^2 - QC^2 = 4SOC$.

Cor. 3. *All these things hold good, if the point* C *is taken without the circle.*

PROP. XXVI.

In a circle, if a perp. DB *be let fall from any point* D, *upon the diameter* CI, *and the tangent* DO *drawn from* D; *then* AB, AC, AO, *will be continually proportional.* 116.

Draw the radius DA, then the triangles ABD, ADO, are similar, for the angles at B and D are right (Cor. 2. 10), and angle A common; whence AB : AD : : AD : AO; that is, AB : AC : AO $\div\div$.

PROP. XXVII.

If a triangle ADC *be inscribed in a circle; and if* BC *be drawn parallel to the tangent* AT; *then* AB, AC, AD, *are continually proportional.* 117.

For the triangle ABC, is similar to ACD; for $\angle$ D = TAC (16) = ACB (4. 11), and A is common; therefore AB : AC : : AC : AD (13. II).

Cor. AD : DC : : AC : CB.

PROP. XXVIII.

If a triangle BDF *be inscribed in a circle, and a perpendicular* DP *let fall from* D *on the opposite side* BF, *and the diameter* DA *drawn; then as the perpendicular, is to one side including the angle* D; *so the other side, to the diameter of the circle*; DP : DB : : DF : DA. 118.

FIG. For drawing AF, the triangles BDP, and ADF
118. are similar; for ∠ A = B (13), and angles at P and F are right (14); therefore DP : DB : : DF : DA (13. II).

Cor. *The rectangles of the sides of an inscribed triangle; is equal to the rectangle of the diameter, and the perp. on the third side.*

PROP. XXIX.

119. *If a triangle* BAC *be inscribed in a circle, and the angle* A *bisected by the right line* AED; *then as one side, to the segment of the bisecting line, within the triangle; so the whole bisecting line, to the other side;* AB : AE : : AD : AC.

Draw BD, then the triangles ABD, ACE are similar; for ∠D = C (Cor. 2. 12), and BAD = EAC (hyp.); therefore AB : AD : : AE : AC (13. II); and AB : AE : : AD : AC (4. Proportion).

Cor. *If an angle of a triangle (inscribed in a circle) be bisected; the rectangle of the sides, is equal to the rectangle of the whole bisecting line within the circle, and the segment within the triangle:* BAC = DAE.

PROP. XXX.

120. *If a circle be inscribed in a triangle* ABC, *and lines be drawn from the center* D, *to the points of contact* E, F, G; *then any segment* BF *or* BE *joining to the angle* B, *is equal to half the sum of the three sides — the opposite side* AC.

For the triangles BDF, BDE are similar and equal (9. II); for ∠F = ∠E a right one (10), and DE = DF, and BD common; whence BF = BE. In like manner CF = CG, and AE = AG. Then since the sum of the sides is BC + CA + AB =

2BF

2BF + 2CG + 2AG, therefore half the ſum = BF + CG + AG = BF + AC, therefore BF = $\frac{1}{2}$ſum — AC. FIG. 120.

Cor. *The area of the triangle* BAC, *is equal to the rectangle of the radius* DF, *and half the ſum of the three ſides.*

For the triangle ABC is made up of the three triangles ADB, BDC, CDA, whoſe common hight is the radius DF.

PROP. XXXI.

If a quadrilateral ABCD *be inſcribed in a circle, the ſum of two oppoſite angles is equal to two right angles;* ADC + ABC = *two right angles.* 121.

Draw AC, BD, and produce AB to E; then the external angle CBE = BCA + BAC (1. II) = BDA + BDC (13) = ADC; therefore CBE + CBA = ADC + CBA = 2 right angles (1. I).

Cor. *If one ſide of a quadrangle (inſcribed in a circle) be produced, the external angle* EBC *is equal to the internal oppoſite angle* ADC.

PROP. XXXII.

If a quadrangle be inſcribed in a circle; the rectangle of the diagonals, is equal to the ſum of the rectangles of the oppoſite ſides; AC × BD = AB × CD + AD × BC. 122.

Make the angle ABF = CBD, then ABD = CBF; and ſince the ∠CDB = FAB (13), the triangles FAB, and CDB are ſimilar, whence DC : DB :: AF : AB (13. II), and CD × AB = BD × AF (12. Proportion). Alſo ſince ∠BCF = BDA (13), the triangles CBF and DBA are ſimilar; whence CB : CF :: DB : DA (13. II), and CB ×

FIG. 122. $\times DA = BD \times CF$ (12. Proportion). Therefore $CD \times AB + CB \times DA = BD \times AF + BD \times CF = BD \times AC$ (Ax. 3).

PROP. XXXIII.

A circle is equal to a triangle whose base is the circumference of the circle; and hight, its radius.

123. Let AB be equal to the length of the circumference, and let the circle touch it in I; draw CI, and CD extremely near it. Then by reason of the extreme smallness of the arch DI, the sector CD coincides with the triangle CDI, and the arch with a portion of the right line. Now since the circle DEGF may be supposed to be made up of such sectors CDI, and the triangle ACB of as many triangles CDI equal to the sector CDI; it follows that all these sectors are equal to all the triangles, or the circle DEGF = the triangle ABC.

This is also evident by the 25. III. for a circle may be considered as a regular polygon of an infinite number of sides, whose hight is the radius of the circle.

Cor. *The sector of a circle is equal to a triangle, whose base is the arch, and hight the radius.*

PROP. XXXIV.

123. *The area of a circle is equal to the rectangle of half the circumference and half the diameter.*

For the circle EGF is equal to the triangle ABC (33), and that triangle is equal to the rectangle of half the base AB and hight CI, that is, of half the circumference DFGED, and half the diameter CI (Cor. 2. 10. II).

Cor. 1. *The sector of a circle is equal to the rectangle of half the arch and the radius.*

Cor.

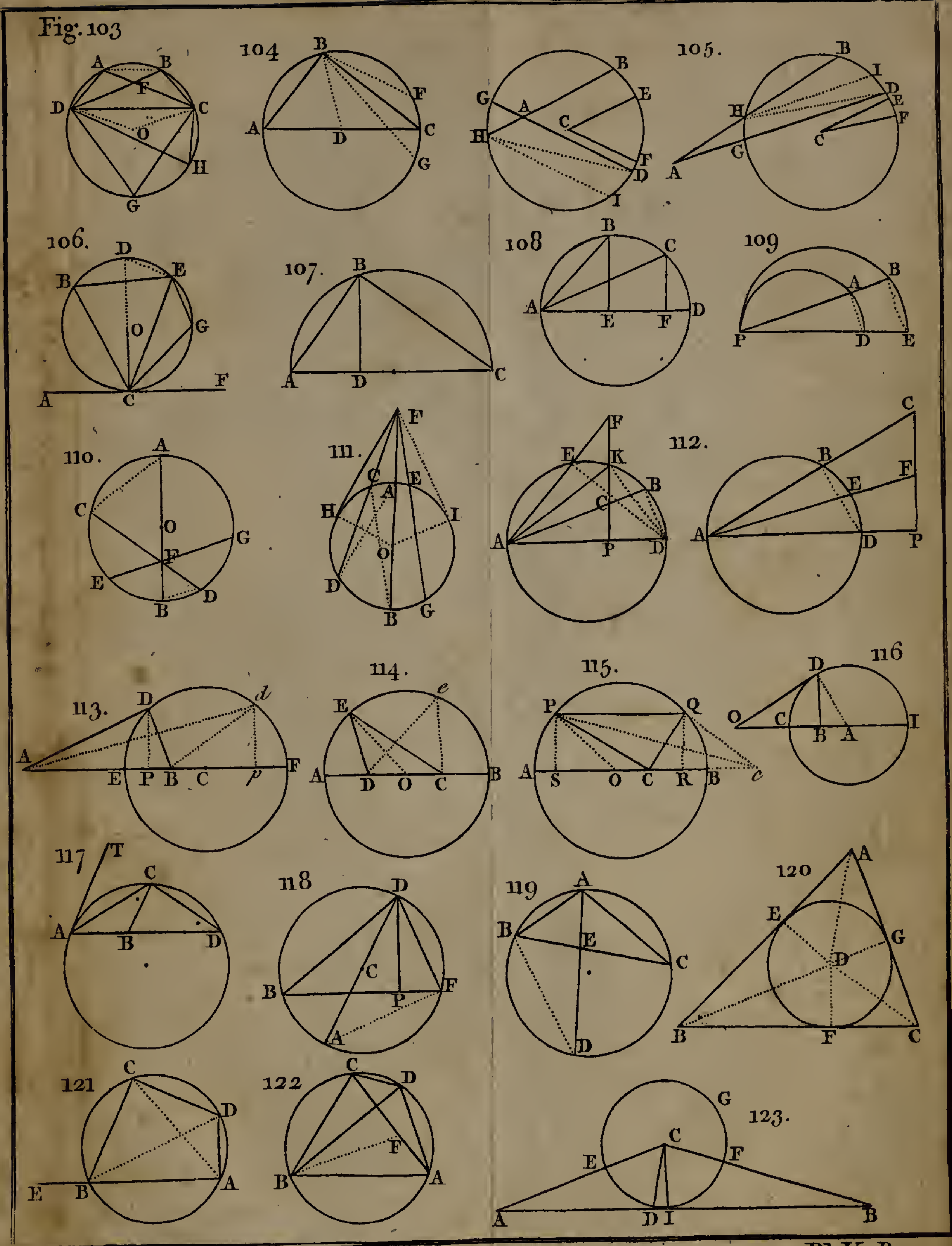
Fig. 103
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105.
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123.

Pl. V. *Pa. 70.*

Cor. 2. *Sectors are to one another in the complicate ratio of their arches and radii.* FIG. 123.

For triangles, to which they are equal, are in that ratio (Cor. 1. 11. II).

PROP. XXXV.

Circles (that is, their areas) are to one another as the squares of their diameters.

For circumference EFGE : cir. IHKI : : AB : CD (9) ; and the areas of the circles EFG, and IHK 124.

are $\frac{EFG \times AB}{4}$, and $\frac{IHK \times CD}{4}$ (34); therefore circle EF : circle IH : : EFGE $\times$ AB : IHKI $\times$ CD (5. Proportion) : : AB^2 : CD^2 (7. Proportion).

Cor. 1. *Circles are to one another as the squares of the radii, or as the squares of the circumferences.*

For these are all in the same ratio (Prop. IX).

Cor. 2. *A circle made on the hypothenuse, is equal to two circles made alike on the two sides, of a right-angled triangle.*

PROP. XXXVI.

Similar polygons described in circles, are to one another, as the circles wherein they are inscribed.

Draw CK, AG, then because similar polygons may be resolved into similar triangles (Cor. 2. 19. III), therefore AF : AG : : CH : CK, and AG : AB : : CK : CD (13. II), therefore AF : AB : : CH : CD. Or at once, AF : AB : : CH : CD (19. III). And polygon EF : polygon IH : : AF^2 : CH^2 : : AB^2 : CD^2 (20. III) : : circle EF : circle IH (35). 124.

FIG. 124. Cor. 1. *Like polygons inſcribed in circles, are as the ſquares of the diameters.*

Cor. 2. *The peripheries of like polygons inſcribed in circles, are as the diameters of the circles.*

For AF : CH : : FG : HR : : GB : KD : : BE : DI : : EA : IC (Def. 12), therefore AFGBEA : CHKDIC : : AF : CH (10. Proportion) : : AB : CD (19. III).

PROP. XXXVII.

A circle is to any circumſcribed rectilineal figure; as the circle's periphery, to the periphery of the figure.

125. From O, the center of the inſcribed circle, draw OP perp. to the ſide AD. Then the figure AC conſiſts of the triangles ABO, BCO, CDO, DAO, whoſe common hight is the radius OP. Therefore its area $= \frac{ABCDA}{2} \times OP$; and the circle = circumference $\frac{PRQ}{2} \times PO$ (34); but $\frac{ABCDA \times OP}{2} : \frac{PRQ \times OP}{2}$: : ABCDA : circumference PQR (5. Proportion).

Cor. 1. *Any polygon circumſcribed about a circle, is equal to a triangle whoſe hight is the radius, and baſe the circumference of the polygon.*

Cor. 2. *Of figures of equal compaſs, the circle is the biggeſt or moſt capacious.*

For if the ſides of the figure be ſuppoſed to touch the circle, they will be greater than the circumference of the circle, contrary to the ſuppoſition. Therefore they will fall within the circle, and then the perpendicular upon them will be ſhorter than the radius. And therefore the polygon will be leſs than the circle, becauſe the triangle (to which it is equal) has

has the fame bafe, and a lefs hight, than the triangle FIG.
to which the circle is equal. 125.

Cor. 3. *All figures circumfcribing the fame circle, are to one another as their circumferences.*

PROP. XXXVIII.

The area of a crown, ring, or annulus ABC *(contained between the circumferences of two circles), is equal to the rectangle under the breadth* RF, *and half the fum of the perimeters.* 126.

Let C, c be the circumferences of the greater and leffer circles, then KF : c : : KR : C (9), and KF : KR : : c : C (4. Proportion), and KF : RF : : c : C — c (13. Proportion), whence $KF \times \overline{C - c} = RF \times c$.

But the annulus = difference of the circles = $\frac{RK \times C}{2} - \frac{KF \times c}{2} = \frac{RF \times C + FK \times C - FK \times c}{2}$ $= \frac{RF \times C + FK \times \overline{C - c}}{2} = \frac{RF \times C + RF \times c}{2}$ $= \frac{C + c}{2} \times RF$.

Cor. *If* FG *be perp. to* RP, *and a mean proportional between the two radii; then the circle defcribed, with the radius* FG, *is equal to the annulus* ABC.

For $FG^2 = KG^2 - KF^2$ (Cor. 1. 21. II.); therefore the circle whofe radius is FG is equal to the difference of the circles whofe radii are KG or KR and KF (22. III).

PROP. XXXIX.

Let ABCD *be a trapezium infcribed in a circle, and put* R = *the radius,* P = AB × BC + CD × DA, Q = AB × CD + BC × AD, T = AB × AD + BC × CD. *Then the area of the trapezium will be* $= \frac{\sqrt{PQT}}{4R}$. 127.

For

FIG. 127. For $\frac{AB \times AD}{2R} =$ perp. from A upon BD (28), and $\frac{BC \times CD}{2R} =$ perp. from C on BD (ibid.), and $\frac{AB \times AD + BC \times CD}{2R} = \frac{T}{2R}$ is the ſum of the perpendiculars.

Therefore $\frac{T \times BD}{2R} =$ twice the area of the trapezium (24. III); and in like manner $\frac{AB \times BC + AD \times DC}{2R} \times AC$ or $\frac{P \times AC}{2R} =$ twice the ſame area. Therefore $T \times BD = P \times AC$, and $AC = \frac{T \times BD}{P}$. But $AC \times BD = AB \times CD + AD \times CB$ (32) $= Q = \frac{T \times BD^2}{P}$, and $BD^2 = \frac{PQ}{T}$, and $BD = \sqrt{\frac{PQ}{T}}$. Whence $\frac{T \times BD}{2R} = \frac{T}{2R}\sqrt{\frac{PQ}{T}} = \frac{\sqrt{PQT}}{2R} =$ twice the area; and the area $= \frac{\sqrt{PQT}}{4R}$.

Cor. 1. $BD^2 = \frac{PQ}{T}$, and $AC^2 = \frac{QT}{P}$.

Cor. 2. $BD : AC :: P : T :: AB \times BC + CD \times DA : AB \times AD + BC \times CD$.

PROP. XL.

127. *If* ABCD *be a trapezium inſcribed in a circle, and each ſide be ſubtracted from the ſum of the other three, there will be four remainders; then take the product of two of theſe remainders, and likewiſe the product of the other two; I ſay, 4 times the area of the trapezium will be a mean proportional, between theſe two products.*

From

From A let fall the perpendiculars AF, AP upon CB, CD. Then since $\angle ABF = ADC$ (Cor. 31); the first perp. falls without, and the second within the figure. Also the angles F, P, being right, the triangles ABF, APD, are similar. Therefore $AB : BF :: AD : DP = \frac{AD \times BF}{AB}$ (13. II), and $AF = \sqrt{AB^2 - FB^2}$ (21. II), also $AB : AF :: AD : AP = \frac{AD}{AB}\sqrt{AB^2 - FB^2}$. Draw AC, then $AB^2 + BC^2 + 2BC \times BF = AC^2 = AD^2 + CD^2 - 2CD \times \frac{AD \times BF}{AB}$ (22, 23. II). Whence $2BC \times BF + \frac{2CD \times AD}{AB} \times BF = AD^2 + CD^2 - AB^2 - BC^2$, and $\frac{BF}{AB} = \frac{AD^2 + CD^2 - AB^2 - BC^2}{2AB \times BC + 2CD \times DA}$ $= \frac{AD^2 + CD^2 - AB^2 - BC^2}{2P}$ (putting $P = AB \times BC + CD \times DA$). And $\frac{AB + BF}{AB} = \frac{CD^2 + 2CD \times AD + AD^2 - AB^2 + 2AB \times BC - BC^2}{2P}$ $= \frac{\overline{CD + AD}^2 - \overline{AB - BC}^2}{2P}$, and $\frac{AB - BF}{AB}$ $= \frac{AB^2 + 2AB \times BC + BC^2 - CD^2 + 2CD \times AD - AD^2}{2P}$ $= \frac{\overline{AB + BC}^2 - \overline{CD - AD}^2}{2P}$.

FIG. 127.

But twice the area $= AF \times BC + AP \times CD$ (24. III) $= BC\sqrt{AB^2 - BF^2} + \frac{CD \times AD}{AB}\sqrt{AB^2 - BF^2}$ $= \frac{AB \times BC + CD \times AD}{AB}\sqrt{AB^2 - BF^2} = \frac{P}{AB}\sqrt{AB^2 - BF^2}$; and $4 \times$ area square $= PP \times \frac{AB^2 - BF^2}{AB^2} = PP \times \frac{AB + BF}{AB} \times \frac{AB - BF}{AB}$ =

FIG. 127. $= PP \times \frac{\overline{CD + AD}^2 - \overline{AB - BC}^2}{2P} \times \frac{\overline{AB + BC}^2 - \overline{CD - AD}^2}{2P}$, and $16 \times \overline{\text{area}}^2 =$

$$\overline{\overline{CD + AD}^2 - \overline{AB - BC}^2} \times \overline{\overline{AB + BC}^2 - \overline{CD - AD}^2}$$
$$= \overline{CD + AD + AB - BC} \times \overline{CD + AD - AB + BC}$$
$$\times \overline{AB + BC + CD - AD} \times \overline{AB + BC + AD - CD}$$
(12. I).

Cor. *If* $S =$ *half the sum of the four sides, then the area is a mean proportional between these rectangles* $\overline{S - AB} \times \overline{S - BC}$, *and* $\overline{S - CD} \times \overline{S - DA}$.

For $\text{area}^2 = \frac{CD + AD + AB - BC}{2} \times$ &c. but $S - AB = \frac{CD + AD + AB + BC}{2} - AB = \frac{CD + AD - AB + BC}{2}$, and so of the rest.

PROP. XLI.

128. *If an equilateral triangle* ABC *be inscribed in a circle; the square of the side thereof, is equal to three times the square of the radius:* $AB^2 = 3AD^2$.

Draw the diameter AE, and the cord BE. Then the triangle BDE is equiangular (Cor. 1. 2. II), for $\angle BDE = BAC$ (Cor. 1. 12) $= BCA = BED$, and $BE = DB$ (Cor. 1. 3. II). Then $AB^2 + BE^2 = AE^2$ (21. II) $= 4DB^2 = 4BE^2$, and $AB^2 = 3BE^2 = 3BD^2$.

Cor. 1. $AB^2 : AF^2 :: 4 : 3$.

For $AB^2 : AF^2 :: AE : AF$ (23. Proportion) $:: 4 : 3$.

Cor. 2. $DF =$ *half* DE.

Cor. 3. *The side* BC *of the equilateral triangle, cuts off a fourth part of the diameter.*

PROP.

PROP. XLII.

FIG. 129.

A square inscribed in a circle, is equal to twice the square of the radius; $AB^2 = 2BO^2$.

For $AB^2 = AO^2 + OB^2$ (21. II) $= 2AO^2$.

Cor. *The circumscribed square* EG *is double the inscribed square*, AC.

For EG is the square of the diameter or 4 squares of the radius, and therefore equal to two of the inscribed squares, ABCD.

PROP. XLIII.

If two diagonals BD, EC *be drawn to cut one another, in an inscribed regular pentagon. The greater segments* EF, BF, *will be equal to the side of the pentagon*, AB. 130.

For since the arch AE = BC, and AB = ED, therefore EC is parallel to AB, and BD parallel to AE (Cor. 13); therefore ABFE is a parallelogram, and EF = AB = AE = BF (1. III).

Cor. 1. *The diagonals* BD, CE *cut one another in extreme and mean proportion*; BD : BF : : BF : FD.

For ∠DCF = CDF = CBD (Cor. 2. 12); therefore the triangles CDF, CDB are similar, BD : DC : : DC : DF (13. II); that is, BD : BF : : BF : FD.

Cor. 2. *The diagonal* CE *is parallel to* AB, *and* BD *to* AE.

Cor. 3. *The side of the pentagon* BC, *is to the diagonal* BD, *as* 1 *to* $\frac{1 + \sqrt{5}}{2}$.

For

FIG. For $BD \times FD$ or $BD \times \overline{BD - BC} = BF^2$ (Cor. 1);
130. that is, $BD^2 - BD \times BC = BC^2$; add $\frac{1}{4}BC^2$, then $BD^2 - BD \times BC + \frac{1}{4}BC^2 = \frac{5}{4}BC^2$; that is, $\overline{BD - \frac{1}{2}BC}^2 = 5 \times \frac{BC^2}{4}$, and the root is $BD - \frac{1}{2}BC = \frac{BC}{2}\sqrt{5}$, and $BD = \frac{BC}{2} + \frac{BC}{2}\sqrt{5} = BC \times \frac{1+\sqrt{5}}{2}$.

Cor. 4. *The angle* BCF *is double to the angle* CBF. For it ſtands on double the arch.

PROP. XLIV.

130. *If a regular pentagon be inſcribed in a circle; the ſquare of the radius* AH, *is to the ſquare of its ſide,* AB; *as* 2 *to* $5 - \sqrt{5}$.

Let HG biſect AB in I, and make $IO = IG$. Then the angles AIO, AIG are right (Cor. 3. 3. II). And put $R =$ radius AH. The $\angle GHA = \frac{2}{5}$ of a right angle (1. I), and $IAH = \frac{3}{5}$ of a right angle (Cor. 2. 2. II); but IAG or $IAO = \frac{1}{2}GHA$ (12) $= \frac{1}{5}$ of a right angle, therefore $OAH = \frac{2}{5}$ of a right angle (Ax. 4) $= OHA$, whence $HO = OA = AG$. $2R \times GI = GA^2$ (Cor. 17) $= HO^2 = \overline{R - 2GI}^2 = RR - 4R \times GI + 4GI^2$ (11. I), and $4GI^2 - 6R \times GI + RR = 0$ (Ax. 3. 4), and $GI^2 - \frac{3}{2}R \times GI + \frac{1}{4}RR = 0$ (Ax. 6); add $\frac{5}{16}RR$, then $\frac{9}{16}RR - \frac{3}{2}R \times GI + GI^2 = \frac{5}{16}RR$; that is, $\overline{\frac{3}{4}R - GI}^2 = 5 \times \frac{RR}{16}$, whence the root $\frac{3}{4}R - GI = \frac{R}{4}\sqrt{5}$, and $GI = \frac{3-\sqrt{5}}{4}R$. But $\frac{3}{2}R \times GI - GI^2 = \frac{1}{4}RR$. And $AI^2 = 2R \times GI - GI^2$ (17) $= \frac{1}{2}R \times GI + \frac{3}{2}R \times GI - GI^2 = \frac{1}{2}R \times GI + \frac{1}{4}RR = \frac{1}{2}R \times \frac{3-\sqrt{5}}{4}R + \frac{RR}{4} =$

$RR \times \frac{5-\sqrt{5}}{8}$, and $4AI^2$ or $AB^2 = RR \times \frac{5-\sqrt{5}}{2}$. FIG. 130.

Cor. 1. *The square of the perpendicular* HI, *upon the side of the pentagon, is equal to* $\frac{3+\sqrt{5}}{8}RR$.

For $HI^2 = R^2 - AI^2 = \frac{8R^2 - 5R^2 + \sqrt{5}\,.\,RR}{8}$ $= \frac{3+\sqrt{5}}{8}RR$.

Cor. 2. *The square of radius* AH, *is to the square of the diagonal* BD, *as* 1 *to* $\frac{5+\sqrt{5}}{2}$.

For BC or $AB = \frac{2BD}{1+\sqrt{5}}$ (Cor. 3. 43), and $AB^2 = \frac{4BD^2}{\overline{1+\sqrt{5}}^2} = \frac{4BD^2}{1+5+2\sqrt{5}}$ (10. I) $= \frac{4BD^2}{6+2\sqrt{5}}$, and $AB^2 = RR \times \frac{5-\sqrt{5}}{2}$ (44), therefore $\frac{4BD^2}{6+2\sqrt{5}}$ or $\frac{2BD^2}{3+\sqrt{5}} = RR \times \frac{5-\sqrt{5}}{2}$, and $2BD^2$ $= RR \times \frac{5-\sqrt{5}}{2} \times \overline{3+\sqrt{5}} = RR \times \frac{15+5\sqrt{5}-3\sqrt{5}-5}{2}$ (Cor. 1. 8. I) $= RR \times \frac{10+2\sqrt{5}}{2}$; that is, BD^2 $= RR \times \frac{5+\sqrt{5}}{2}$.

Cor. 3. *If* CE *be the diagonal of the pentagon, and* OLD *be drawn; then* $DL = R \times \frac{5-\sqrt{5}}{4}$. 134.

For $DL = \frac{DE^2}{DF}$ (Cor. 17) $= RR \times \frac{5-\sqrt{5}}{4R} =$ $R \times \frac{5-\sqrt{5}}{4}$ (44).

PROP.

FIG.

PROP. XLV.

131. *The ſide of a regular hexagon inſcribed in a circle, is equal to the radius of the circle:* BE = BD.

For $\angle$BDE = $\frac{1}{6}$ of four right angles (Cor. 1. I) = $\frac{1}{3}$ of two right angles. And the angles B and E together = $\frac{2}{3}$ of two right angles (2. II), whence BED = $\frac{1}{3}$ of two right angles (3. II) = BDE; therefore BE = BD (Cor. 1. 3. II).

PROP. XLVI.

132. *The ſquare of the ſide of a regular octagon, inſcribed in a circle; is equal to the ſquare of half the ſide of the inſcribed ſquare, together with the ſquare of the difference of that half ſide and the radius;* $AB^2 = AP^2 + \overline{OB - AP}^2$.

For $AB^2 = AP^2 + PB^2$ (21. II), but $\angle$PAO = POA (Cor. 1. 12); therefore PO = AP, and BP = OB — OP = OB — AP; and $AB^2 = \overline{OB - AP}^2 + AP^2$.

Cor. *The ſquare of radius, is to the ſquare of the ſide of the octagon; as* 1 *to* $2 - \sqrt{2}$.

For $AP = \sqrt{\frac{1}{2}AO^2}$, and $PB^2 = \overline{BO - AP}^2 = \overline{BO - AO\sqrt{\frac{1}{2}}}^2 = BO^2 - \frac{2BO \times AO}{\sqrt{2}} + \frac{AO^2}{2} = 1\frac{1}{2}AO^2 - AO^2\sqrt{2}$, add $AP^2 = \frac{1}{2}AO^2$, and $AP^2 + BP^2$ or $AB^2 = 2AO^2 - AO^2\sqrt{2}$.

PROP. XLVII.

133. *The radius of a circle is a mean proportional, between the ſide of an inſcribed regular decagon, and the ſum of that ſide and the radius;* AB : DA :: DA : DA + AB.

Pro-

Produce AB to F, so that BF may be = BD; FIG. 133.
and draw DF, DB. Then $\angle ADB = \frac{1}{5}$ of two right angles, and therefore DAB and DBA together $= \frac{4}{5}$ of two right angles (2. II), and ABD $= \frac{2}{5}$ of two right angles (3. II) $= BDF + BFD$ (1. II) $= 2BDF$ (3. II); therefore BDF or BFD $= \frac{1}{5}$ of a right angle $= ADB$. Therefore the triangles ADB, and ADF are similar, for F = ADB; and A is common, whence AF or AB + BD : AD : : AD : AB.

Cor. 1. *If the radius be cut in extreme and mean ratio, the greater segment is the side of the decagon,* AB.

For since AB + AD : AD : : AD : AB. Therefore AB : AD : : AD — AB : AB (13. Proportion), or AD : AB : : AB : AD — AB, therefore AD is cut in extreme and mean proportion (Def. 11. Proportion).

Cor. 2. *The radius is to the side of the decagon; as* 2 *to* $\sqrt{5} - 1$.

For $AB^2 + AB \times AD = AD^2$ (12. Proportion), add $\frac{1}{4}AD^2$, then $AB^2 + AB \times AD + \frac{1}{4}AD^2 = \frac{5}{4}AD^2$ (Ax. 3), and $AB + \frac{1}{2}AD = \frac{AD}{2}\sqrt{5}$, and $AB = \frac{AD}{2}\sqrt{5} - \frac{AD}{2}$, and $2AB = AD \times \overline{\sqrt{5} - 1}$.

Cor. 3. *The square of the perpendicular upon the side of a decagon, is* $\frac{5 + \sqrt{5}}{8}$ × *the square of the radius.*

For $\frac{1}{2}AB$ = rad. $\times \frac{\sqrt{5} - 1}{4}$, and its square =

FIG. 133. $AD^2 \times \frac{3-\sqrt{5}}{8}$, and the ſquare of the perpend. $= AD^2 - AD^2 \times \frac{3-\sqrt{5}}{8} = AD^2 \times \frac{5+\sqrt{5}}{8}$.

PROP. XLVIII.

134. *The ſquare of the ſide of a regular pentagon inſcribed in a circle, is equal to the ſum of the ſquares of the radius, and of the ſide of a regular decagon, inſcribed in the ſame circle*; $AB^2 = FA^2 + AO^2$.

Draw OG perpendicular to the cord FA, to cut it in G, and draw FH. The triangles ABO, HBO are ſimilar; for $\angle AOB = \frac{1}{5}$ of 4 right angles, or $\frac{2}{5}$ of 2 right angles (1. I), alſo BAO and ABO are together $= \frac{3}{5}$ of 2 right angles (2. II), and therefore BAO $= \frac{3}{10}$ of 2 right angles, but BOG ($= \frac{3}{4}$ AOB) $= \frac{3}{4} \times \frac{2}{5}$ of 2 right angles $= \frac{3}{10}$ of 2 right angles, therefore BAO = BOG, and B is common; whence AB : BO : : BO : BH $= \frac{BO^2}{AB}$.

Again, the triangles AFH and ABF are ſimilar, for $\angle A =$ AFH (3. II), and $\angle A =$ B (Cor. 2. 12), therefore BA : AF : : AF : AH $= \frac{AF^2}{AB}$; therefore $AB = AH + BH = \frac{AF^2 + BO^2}{AB}$, or $AB^2 = BO^2 + AF^2$.

Cor. *The perpendicular* OI *upon the ſide of the pentagon, is equal to half the ſum of the radius and ſide of the decagon*; $OI = \frac{AO + AF}{2}$.

For $OI = OF - FI = \frac{2OF^2 - 2OF \times FI}{2OF} = \frac{2OF^2 - FA^2}{2OF}$ (Cor. 17). And ſince $AO^2 = FA^2 +$

AO

$AO \times AF$ (47), therefore $AO^2 - FA^2 = AO \times$ FIG.
AF, and $2FO^2 - FA^2 = FO^2 + AF \times FO$, and 134.
$\frac{AF + FO}{2} = \frac{2OF^2 - FA^2}{2OF} = OI$.

PROP. XLIX.

The side of a regular dodecagon inscribed in a circle, is a mean proportional between the radius, and the excess of the diameter above the side of the inscribed equilateral triangle. 135.

Let AB be a side of the dodecagon, and draw CB, CF, and DF the side of the triangle, and FR perpendicular to AC. Then $ACF = \frac{1}{3}$ of 2 right angles $= CAF = CFA$ (2. II), therefore ACF is an equilateral triangle, and $AO = \frac{1}{2}AC$, and $CO = \sqrt{\frac{3}{4}AC^2}$ (Cor. 39. II), and $BO = CA - CO = CA - \sqrt{\frac{3}{4}CA^2}$, and $BO^2 = CA^2 + \frac{3}{4}CA^2 - 2CA \times \sqrt{\frac{3}{4}CA^2} = 1\frac{3}{4}CA^2 - CA^2\sqrt{3}$ (11. I), and $AB^2 = AO^2 + OB^2$ (21. III) $= \frac{1}{4}AC^2 + 1\frac{3}{4}AC^2 - CA^2\sqrt{3} = 2AC^2 - CA^2\sqrt{3}$. Therefore $CA : AB :: AB : 2CA - \overline{CA\sqrt{3}}$. But 2CA is the diameter, and $CA \times \sqrt{3} =$ side DF of the equilateral triangle (41).

Cor. *The side of the dodecagon,* $AB = CA \times \sqrt{2 - \sqrt{3}}$.

PROP. L.

If ABH *be an equilateral triangle, and* APDFG *an equilateral pentagon, inscribed in a circle, both placed with their angles at* A; *then the cord* BD *will be the side of an equilateral quindecagon; and* BI *will be half the difference of the sides of the triangle and pentagon, and* DI (*perp. to* BH) *is the difference of the perpendiculars in the two figures; and* $BD^2 = BI^2 + DI^2$. 136.

For

FIG. 136. For arch AP $= \frac{1}{5} = \frac{3}{15}$ the circumference, and APD $= \frac{6}{15}$ of it; alſo APB $= \frac{1}{3} = \frac{5}{15}$ the circumference; therefore APD — APB $= \frac{1}{15}$ the circumference; and the cord BD, the ſide of the quindecagon; moreover DI $=$ CO — CE, is known from Cor. Prop. XLIV. and Cor. 2. Prop. XLI. Alſo BI $= \frac{BH - DF}{2}$, which will be known by Prop. XLI, and Prop. XLIV; and thence $BD^2 = BI^2 + DI^2$, will be known.

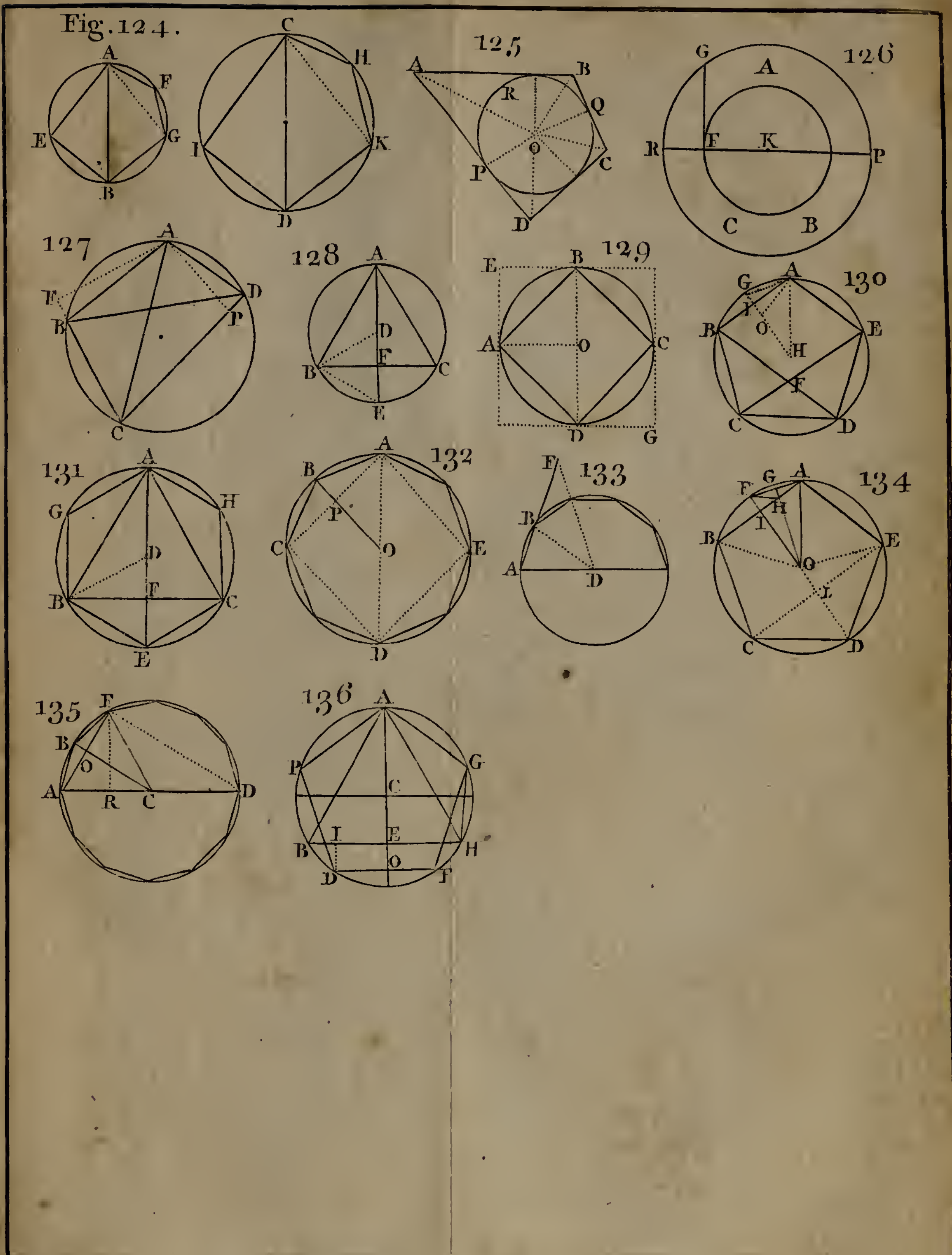
Fig. 124.
A F G B E
C H K D I
125
A B R Q O C P D
126
G A F K R P C B
127
A D F P B C
128
A D F B C E
129
E B A O C D G
130
G A I O B E H F C D
131
A G H D B F C E
132
B A P C O E D
133
E B A D
134
F G A H I B E O L C D
135
E B O A R C D
136
A P G C B I E H O D F

Pl.VI. *Pa.* 84.

BOOK V.

Of Planes, and ſolid Angles.

DEFINITIONS.

1. A *Plane* is a ſurface which lies even between the extremes; or in which all right lines coincide. FIG.

2. A *curve ſurface*, is that whoſe parts lie not even between the extremes; but gradually riſe or fall.

3. A *convex ſurface*, is that which ſwells or riſes up towards the middle.

4. A *concave ſurface*, is that whoſe middle parts are hollow, or fall lower than the extremes.

5. A *right line*, is ſaid to be *perpendicular* to a plane, when it is perpendicular to all lines, drawn to the foot of it, as AB. 140.

6. The *common ſection* of two planes, is the line made by two planes cuting each other, as EF. 139.

7. One *plane* is ſaid to be *perpendicular* to another, when it paſſes through a right line which is perpendicular to the other, as CD. 144.

8. The *inclination* of a *line* to a plane, or the angle it makes with it, is the angle that line makes with a line drawn from the foot of it to the point where a perpendicular, let fall from the top, cuts the plane, as FCI. 142.

9. The *inclination* of two planes, is the angle made by two right lines; both drawn perpendicular to the common ſection, from any point therein; 149.

 in;

FIG. in; as ∠BFD is the inclination of the planes,
149. AB, CD.

This is the angle which the planes make with one another.

145. 10. *Parallel planes* are those which are every where at the same distance from each other, as AC, EG.

150. 11. A *solid angle* is a space bounded by several plane angles, meeting in one point, called the *angular point or vertex*, as A.

PROP. I.

137. *If a right line in a plane be produced, it will still be in that plane.*

For produce BC in the plane AD, directly to E; and if BCF be also a right line, then BCE and BCF are both right lines; and you have two right lines BCF, BCE, with the part BC the same in both; contrary to the nature of a right line (Def. 5. I).

Cor. 2. *If two distant points of a right line be in a plane, all the line is so.*

PROP. II.

138. *If two right lines* GH, IK, *intersect one another, they are in the same plane.*

For imagine A, and B, to be in one plane with C; then the line IACK, as also HBCG, and all the points between A and B, are in one plane (Prop. I. and Cor.). And the like for all other lines as AB, drawn between GCH, and ICK.

Cor. *Every part of a triangle is in the same plane.*

PROP. III. FIG.

If two planes AB, CD, *cut one another; their common section* EF *is a right line.* 139.

For draw the line EF in the plane AB, then any point G of that right line, is in the plane AB (1); but becaufe the line EF (drawn between the two points E, F, in the plane CD) is alfo in the plane CD; any point G of that line will be in the plane CD. Therefore G being in both planes, will be in their common fection; and their common fection EGF is confequently a right line.

PROP. IV.

If a right line AB *be perpendicular to two lines* IK, GH, *at the point of interfection* B; *then is the line* AB *perpendicular to the plane* FD *paffing through them.* 140.

Let BI = BG = BK = BH, and draw GI, HK, LM, AI, AL, AG, AH, AM, AK. The triangles ABI, ABG, ABH, ABK, are all equal (6. II), and AI = AG = AH = AK. Alfo the triangles GBI, HBK are equal (6. II), and GI = KH, ∠G = ∠H. Alfo the triangles LBG, MBH are equal (7. II), and BL = BM, and GL = HM. And the triangles AGI, AKH are equal (8. II); and confequently ∠AGI = ∠AHM; whence the triangles AGL, AHM are equal (6. II), and AL = AM. Alfo the triangles ABL, ABM are equal (8. II), and ∠ABL = ABM = a right angle. And therefore AB is perp. to LM.

Cor. *If a right line* AB *be perpendicular to feveral lines meeting in* B, *as* IB, LB, GB; *thefe lines are all in one plane.*

FIG. 140. For if any of them was out of the plane, AB would make an angle with it, greater or lesser than a right angle.

PROP. V.

141. *Two right lines,* AB, CD, *perpendicular to a plane, are parallel.*

Make BDI a right angle, and DI = AB, and draw BI, AI, AD; AB is ⊥ to BD (Def. 5); and ∠ ABD = BDI; therefore the triangles ABD, DBI are equal (6. II), and AD = BI. Then the triangles ADI and ABI are equal (8. II); and ABI = ADI = a right angle. Therefore ID is ⊥ to DC, DA, DB; and therefore all three are in one plane (Cor. 4). Therefore AB, CD are in the same plane (Cor. Prop. II), and are likewise parallel (Cor. 3. 4. I).

Cor. 1. *Two parallel lines* AB, CD, *are in the same plane.*

Cor. 2. *A line drawn from one parallel to another, is in the same plane with them.*
By Cor. 1, and Cor. to Prop. I.

Cor. 3. *Through one point, there can be drawn but one line perpendicular to a plane.*

PROP. VI.

141. *If one* AB, *of two parallels, be perpendicular to a plane; the other will also be perpendicular to it.*

Suppose as in the last Prop. then the angles IDA, IDB, are right. Therefore DI is ⊥ to the plane ADB, in which AB, CD are (Cor. 1. 5); there-

therefore ID is ⊥ to CD; but CDB = a right angle. FIG.
Therefore CD is ⊥ to the plane EG. 141.

PROP. VII.

If FI *be perpendicular to the plane* DE, *and* FC *perpendicular to a line* AB, *drawn in that plane; then the line* CI *joining their sections, is also perpendicular to the line* AB. 142.

For first, suppose CB ⊥ to CI, draw IG parallel to CB, then IG being ⊥ to CI and FI, is ⊥ to the plane CFI (4); and ACB is also ⊥ to the plane FCI (6); therefore BC is ⊥ to CI and to CF. And on the contrary being ⊥ to CF, it is also ⊥ to CI; otherwise it could not be ⊥ to the plane FCI; nor its parallel GI.

PROP. VIII.

Right lines AH, CI, *parallel to the same right line* EG, *though not in the same plane, are parallel to one another.* 143.

In the plane of the parallels AH, EG, let HG be ⊥ to EG. Also in the plane of the parallels EG, CI, draw GI ⊥ to EG. Therefore EG is ⊥ to the plane HGI; therefore AH, CI are also ⊥ to the same plane HGI (6), whence AH and CI are parallels (5).

PROP. IX.

If two planes AB, CD *be perpendicular to one another; and from any point* P *in one, a perpendicular* PN, *be let fall to the other; it shall fall upon the common section* FI. 144.

For the line PN ⊥ to the common section, is ⊥ to the plane AB (Def. 7), and if another perp. could be drawn which falls not upon the common section;

FIG. section; then two perpendiculars might be let fall
144. from one point, which is absurd (Cor. 3. 5).

Cor. *A line* NP *in one plane, perpendicular to the common section of two perp. planes, will be perp. to the other plane.*

PROP. X.

145. *Those planes* AC, EG, *are parallel, when the same right line* IK, *is perpendicular to both.*

Draw DL parallel to IK, and draw ID, KL; then since the angles LKI = KID (hyp.) = IDL (6) = a right angle, therefore KLD is a right angle (16. III); therefore ID is parallel to KL (Cor. 3. 4. I); whence IKLD is a parallelogram, and IK = DL, therefore AC is parallel to EG (Def. 10).

Cor. *If a right line is perpendicular to one of two parallel planes, it is perpendicular to the other.*

PROP. XI.

145. *If two parallel planes* AC, EG, *be cut by a third* IL; *their common sections are parallel*; ID, *and* KL.

For it was proved in the last prop. that IDLK is a parallelogram, and that ID and KL are parallel.

Or thus.

Let the plane IBED cut the parallel planes AC, EG, in the sections ID, BE. Now if ID, BE be not parallel, or equidistant, they will meet some way; and consequently the planes wherein they are placed, must meet, which is absurd.

Cor. *If a line* ID *be parallel to the plane* EG; *all planes drawn through this line* ID, *shall intersect the plane* EG *in lines parallel to* ID, *and to one another.*

For

For KL is parallel to ID, and BE is parallel to ID, and therefore KL, BE are parallel to one another (8). FIG. 145.

PROP. XII.

Right lines AQ, BR, *cut by parallel planes*, G, H, I, *are cut proportionally*; AC : CE : : BD : DF. 146.

Draw AB, EF; and BE to cut the plane H in P. Then in the planes, BEF, EAB, the sections PD, EF, as also CP, AB, will be parallel (11); therefore in the triangles BEF, EAB; AC : CE : : BP : PE : : BD : DF (12. II).

Cor. *The segments of parallel lines, cut off by parallel planes, are equal.*

PROP. XIII.

If two lines AB, AC, *cuting one another, be parallel to two other right lines*, ED, DF, *cuting one another, though not in the same plane; these lines will make equal angles*; BAC = EAD. 147.

Let AB = DE, AC = DF, and draw BE, AD, CF, and also BC, EF. Since AB, DE are parallel and equal, therefore BE, AD are equal and parallel (Cor. 3. 5. I). For the same reason CF, AD, are equal and parallel. Therefore BE, FC are parallel and equal (Prop. VIII. and Ax. 1). Therefore BC is equal and parallel to EF (Cor. 3. 5. I). The triangles BAC, EDF, have all their sides equal, therefore ∠BAC = EDF (8. II).

PROP. XIV.

If two lines AB, AC, *which meet one another, be parallel to two other lines* DE, DF, *that also meet one another; though not in the same plane; the planes* BC, EF, *drawn through them, will be parallel.* 148.

Let

FIG. 148. Let AG be perpendicular to the plane EF, and GH, GI parallel to DE, DF; then GH, GI will be parallel to AB, AC. And since IGA, HGA are right angles, CAG, BAG, will be right angles (4. I); therefore GA is ⊥ to the plane BC, and since it is ⊥ to the plane EF (construct.), therefore the planes BC, EF are parallel (10).

PROP. XV.

149. *If two planes* AB, CD, *which cut one another, be both of them perpendicular to a third plane* GH; *their common section* EF, *shall also be perpendicular to the third plane*, GH.

For a perpendicular to the plane GH, at the point F (in the common section of the planes AB, GH), must be somewhere in the plane AB (Def. 7). Also a perpendicular at F (in the common section of the planes CD, GH), must be somewhere in the plane CD (ibid.); therefore it must be in their common section; that is, the common section EF is ⊥ to the plane GH.

Cor. *The common section* EF *will be perpendicular to* FD, *or* FB, *the section of either plane with the third.*

PROP. XVI.

150. *In a solid angle* A, *contained under three plane ones,* BAD, DAC, BAC; *any two of them is greater than the third.*

Let BAC be the greatest, and let ∠ BAE = BAD, and AD = AE. And draw BEC, BD, DC. The triangle BAE = BAD, for BA, AE are equal to BA, AD, and ∠BAE = BAD, therefore BE = BD, and AE = AD (6. II). But BD + DC is greater than BC (5. II), and DC greater than EC. And since AD = AE, and AC common,

common, ∠CAD is greater than CAE (Cor. 6. II). FIG.
Therefore BAD + CAD is greater than BAC. 150.

PROP. XVII.

Every solid angle is contained under less plane angles than four right angles. 151.

Suppose a plane to cut the sides of the angle, and to make a polygon BCDE, to consist of as many triangles, as there are to make up the solid angle A. Let X = sum of all the internal angles of the polygon B, C, D, *&c.* Y = sum of all the angles at the bases of the triangles composing the solid angle, EBA, ABC, *&c.* Then will X + 4 right angles = Y + A (2. II). But since EBA + ABC is greater than B (16), *&c.* therefore Y is greater than X, and consequently A is less than 4 right angles.

PROP. XVIII.

These solid angles are equal A, G; *which are contained under the same number of plane angles, alike situated, and having the same inclinations, respectively.* 151. 152.

For since ∠BAC = HGI; CAD = IGK, *&c.* therefore if HGI be laid upon BAC, they will coincide, and GI will fall upon AC. Also if IGK be laid upon CAD, they will likewise coincide. And moreover, since the inclination of the planes HGI and KGI is the same as BAC and DAC; therefore the solid under HGIK will exactly coincide with that under BACD. For the same reason the solid, under the planes IGKL and CADE, will likewise concide; and also the solid under KGLH and DAEB will coincide; and those under LGHI, and EAEC, will coincide; and so the whole solid angle G will coincide with the whole solid angle A, and consequently they are equal (Ax. 8).

PROP. XIX.

FIG. 153. 154. *If two solid angles A, B, be contained under three plane angles respectively equal, and alike situated; the like planes have the same inclination to one another.*

Let ∠KAD = MBG, KAE = MBH, and EAD = HBG; the ∠ made by KAD and KAL, will be equal to that made by MBG and MBN. For make BM = AK, and let KD, KL be ⊥ to AK, and MG, MN ⊥ to BM. Draw LD, NG; in the triangles KAD, MBG, ∠ KAD = MBG, and K, M right, and AK = BM; therefore KD = MG, and AD = BG (7. II). For the same reason, in the triangles KAL, MBN; KL = MN, and AL = BN. And in the triangles LAD, NBG; LA, AD are equal to NB, BG, and ∠ A = B, therefore LD = NG (6. II). In the triangles KLD, MNG; the three sides are equal; therefore ∠DKL = ∠GMN, which are the inclination of the planes. And the same way it is demonstrated for the other planes.

Cor. *These solid angles are equal, which are contained under three plane angles, respectively equal.*

For the planes of these angles will have the same inclination to one another respectively (19); and consequently the solid angles, contained thereby, will be equal (18).

SCHOLIUM.

It is evident from hence, that a solid angle, consisting of 3 planes, is determined from the quantity
153. of the 3 plane angles it consists of. For (fig. 153), the triangle KLD, which is its base, is determined from the three sides, KL, LD, KD, being given. And if the point A be also given; the planes AKL, ALD, AKD, are capable of no alteration in their position;

pofition; and fo the folid angle A is determined. FIG.
But although a folid angle of 3 plane angles is determined from the quantity of the angles alone; yet when 4 or more planes are concerned, the quantity of their angles is not fufficient. This will be plain by infpecting fig. 155. Where the bafe of the folid angle A, is the trapezium BCDI. 155. For the 4 fides of the trapezium alone are not fufficient to determine its figure; and by altering its figure, the pofition of the planes is altered (though the feveral angles are not), and confequently the quantity of the folid angle A, is altered. So that the folid angle can no more be determined, from the plane angles given; than a trapezium can, by having all its fides given; and much lefs can it be fo in polygonal angles and bafes.

PROP. XX.

If there be two folid angles A, G, *and the fides of one,* AB, AC, AD, AE, *be refpectively parallel to the fides* GH, GI, GK, GL, *of the other; thefe folid angles will be equal.* 151. 152.

For fince AB, AC, are parallel to GH, GI; $\angle$BAC = HGI (13); for the fame reafon $\angle$CAD = IGK, DAE = KGL, EAB = LGH. Moreover, as AB, AD are parallel to GH, GK; $\angle$BAD = HGK, therefore the folid angle made by the three planes BAC, CAD, BAD, is equal to that made by the three planes HGI, IGK and HGK (Cor. 19). For the fame reafon the folid angle made by the three planes CAD, DAE, CAE is equal to that made by IGK, KGL, IGL. And for the fame reafon the folid angle A made by DAE, EAB = folid angle G made by KGL, LGH. And folid angle made by EAB, BAC = folid angle made by LGH, HGI. Whence all the parts of the folid angles A, G, being mutually equal, and having a like

FIG. 151. 152. like ſituation; the whole angle A, muſt be equal to the whole angle B.

Cor. *In two ſolid angles* A, G, *whoſe planes* BAC, CAD, *&c. are reſpectively parallel to the planes* HGI, IGK, *&c. theſe ſolid angles will be equal.*

For it comes to the ſame thing, whether the lines AB, GH, be parallel, or the planes BAC, HGI, *&c.* (14).

BOOK

BOOK VI.

Of Solids.

DEFINITIONS.

1. A *Pyramid*, is a solid ABD, made by the motion of a line as AB, along the circumference BCDIB of the plane figure BD, the other end at A, remaining fixt. The figure BCDI is called the *base* of the pyramid. The fixt point A is the *vertex*. If the base be a triangle, it is a *triangular pyramid*; if a polygon, a *multangular pyramid*. FIG. 155. 150.

2. A *cone* is a solid generated by a line AB moving about the circle BCD, the end A remaining fixt. The *vertex* is the fixt point A. The *axis* is the line AO drawn from the vertex to the center O of the circle. The *base* is the circle BCD. The *side* is AB or AD. It is called a *right cone*, if the axis is perpendicular to the base; otherwise an *oblique* or *scalene cone*. An *equilateral cone*, is a right cone whose side is equal to the diameter of the base. 156.

3. A *cylinder* is a solid, formed by a line FB moving about two equal and parallel circles, so as that the moving line always keep parallel to the line PO joining their centers. The circle FG or BD is called the *base*. The line PO, drawn between the centers of the circles, is the *axis*. If the axis is perpendicular to the base, it is a *right cylinder*; if not, an *oblique one*. FB or GD is the *side*. If the side of a right cylinder be equal to the diameter of the base, it is called an *equilateral cylinder*. 157.

FIG. 4. A *prism* is a solid, as ACEH, whose ends are
159. two similar equal plane figures, and parallel to one another; and the sides, are parallelograms. The *base* is the plane figure at either end ABCD or HGEF. If all the sides are perpendicular to the base, it is a *right prism*; otherwise an *oblique one.*
158. If the base is a triangle, it is a *triangular prism*; if a polygon, a *multangular prism.*

Cor. *A cylinder is a prism of an infinite number of sides.*

160. 5. A *parallelopipedon* is a prism contained under six plane figures, whose bases, and opposite sides are parallel, as ABD. If the sides are all perpendicular to the bases, it is an *upright parallelopipedon*; if not, an *oblique one.*

161. 6. A *cube* is a solid contained under six equal squares, set perpendicular to one another, as AB.

7. A *polyedron*, is a solid contained under several rectilineal figures.

8. A *regular solid* or *body*, is a solid contained under some number of equal and regular plane figures of the same sort; otherwise, they are *irregular bodies.*

160. 9. *Hight*, of a solid, is the perpendicular falling from the vertex or top, upon the base, as BP.

10. *Frustum*, of a solid, is the lower part, cut off by a plane parallel to the base.

11. *Similar pyramids*, are those contained under similar plane figures, equal in number, and alike placed.

12. *Similar solids* are those which are made up of an equal number of similar pyramids, alike placed: or which may be resolved into such.

13. *Area*, is the quantity of the superficies of any plane figure.

14. *Bodies* are said to *touch* one another, when they meet, but do not cut or enter into one another.

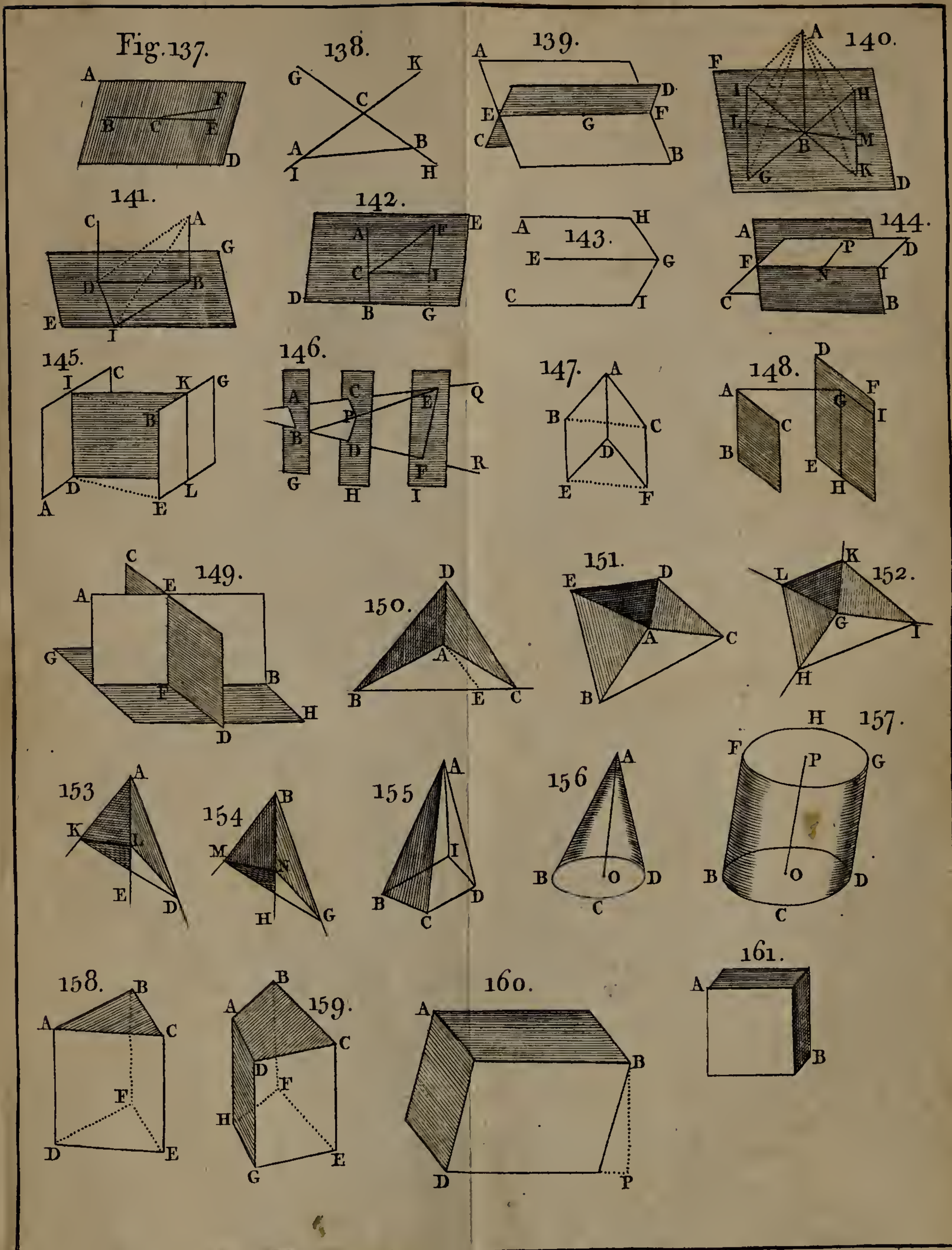
Fig. 137.
138.
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PROP. I.

FIG. 162.

In any parallelopipedon EH, *the oppoſite planes* AE, HB, *are ſimilar and equal parallelograms.*

The plane AC, cuting the parallel planes AG, DB, make the ſections AH, DC parallel (11. V). And the ſame plane AC, cuting the parallel planes AE, HB, make the ſections AD, HC parallels (ibid.); therefore ADCH is a parallelogram. By the ſame reaſoning, all the other planes are parallelograms. Therefore BG = CH = DA = EF (1. III). And ſince DA, AF are parallel to CH, HG; therefore ∠DAF = CHG (13. V); therefore the parallelogram AE = HB (Ax. 8), having equal ſides and angles. And the ſame way it is ſhewn of the other oppoſite planes.

PROP. II.

163.

If a priſm HC, *be cut by a plane parallel to the baſe* AC; *the ſection* EG, *will be ſimilar and equal to the baſe.*

Since AE, BF, CG, DI are parallel (Def. 4), and the plane ABFE is cut by the parallel planes AC, EG, the ſections AB, EF will be parallel, therefore ABFE is a parallelogram, and EF = AB (1. III). For the ſame reaſon FG = BC; GI = CD, and EI = AD. And ſince AB, BC are parallel to EF, FG; ∠ABC = EFG (13. 5). After the ſame manner ∠C = G, and I = D, and E = A. Whence the figure EFGI is ſimilar and equal to ABCD (Ax. 8), having the ſides and angles all equal.

Cor. *If a parallelopipedon be cut by a plane parallel to any ſide; the ſection will be ſimilar and equal to that ſide.*

FIG. 163. For in that ſolid, any ſide may be taken for the baſe (1).

PROP. III.

The ſurface of any polyedron, is equal to the ſum of the areas of all the figures that incloſe it.

For all theſe figures make up the ſurface, therefore the ſum of their areas is equal to the area of the whole (Ax. 2).

Cor. 1. *The ſurface of a pyramid is equal to the ſum of all the triangles incloſing it, together with the baſe.*

159. Cor. 2. *The ſurface of an upright priſm* AE, *is equal to the rectangle of its hight,* CE, *and the circumference of its baſe,* GEFH.

For all the ſides are rectangles of the ſame hight, all which are equal to a rectangle, whoſe baſe is the ſum of all theſe, and hight the ſame (8. III).

Cor. 3. *The ſurface of any regular body is equal to the area of one of the faces, multiplied by the number of them.*

PROP. IV.

164. *The curve ſurface of a right cylinder* AD, *is equal to the rectangle of its hight, and the circumference of the baſe:* BD × CKDC.

Suppoſe FK, OI to be drawn upon its ſurface parallel to the axis, and extremely near together. Then the part of the ſurface OK is equal to the ſmall parallelogram OIKF, or OI × IK (Cor. 2. 6. III). In like manner the whole ſurface may be divided into ſuch parallelograms, the ſum of all which, will be = the ſum of all the IK's × OI; that is, the curve ſurface will be = the circumference CKDC × OI.

Cor.

Cor. 1. *The curve surface of a right cylinder, is equal to a circle whose radius is a mean proportional between the side* AC, *and diameter of the base* CD. FIG. 164.

For let R, C be the radius and circumference of this circle, A its area. Then AC : R : : R : CD (hyp.), and AC : $\frac{1}{2}$R : : 2R : CD (Cor. 3. 12. Proportion) : : C : circumference CKDC (9. IV). Therefore AC $\times$ CKDC = $\frac{1}{2}$RC; that is, the surface of the cylinder = A (34. IV).

Cor. 2. *As half the radius of the base : to the side : : so the base of the cylinder : to its curve surface.*

For the base $= \frac{CD \times CKDC}{4}$ (34. IV), and $\frac{CD \times CKDC}{4}$: surface AC $\times$ CKDC : : $\frac{CD}{4}$: AC (Cor. 1. 5. Proportion).

Cor. 3. *The curve surfaces of right cylinders, are in the complicate ratio of the hights, and diameters of the bases.*

For their equal rectangles are in that ratio (Cor. 2. 8. III), and the diameters are as the circumferences (9. IV).

PROP. V.

The curve surface of a right cone, is equal to the area of a triangle, whose hight is the side AB, *and base the circumference of the cone's base*, BKDB. 165.

Take the very small arch IK, and draw AI, AK. Then the part of the surface AIK coincides with the small isosceles triangle AIK, whose base is IK, and hight AI. In like manner the whole curve surface of the cone, may be supposed to consist of such triangles, whose common hight is AI, and bases so many KI's. All which triangles are equal to the triangle whose

 hight

FIG. 165. hight is AI; and base, the sum of all the IK's, or the circumference BKDB (Ax. 2).

Cor. 1. *The curve surface of a right cone is equal to half the rectangle of the side* AB, *and circumference of the base*, BKDB.

For half of that rectangle is equal to a triangle of the same hight and base (7. III).

Cor. 2. *The curve surface of a right cone is equal to a circle, whose radius is a mean proportional between the side* AB, *and the radius of the base* BC.

For the conic surface $= \frac{AB \times BKDB}{2}$, and the area of the base BD $= \frac{BC \times BKDB}{2}$ (34, IV). Let the radius R $= \sqrt{AB \times BC}$, its area $=$ A. Then conic surface : circle BD : : $\frac{AB \times BKDB}{2}$: $\frac{BC \times BKDB}{2}$: : AB : BC (Cor. 1. 5 Proportion) : : AB $\times$ BC : BC^2 (5. ibid.).

And circle BD : circle A : : BC^2 : R^2 or AB $\times$ BC (Cor. 35. IV). Therefore conic surface : circle A : : AB $\times$ BC : AB $\times$ BC (15 Proportion). Therefore the conic surface $=$ circle A, whose radius is $\sqrt{AB \times BC}$ (Ax. 7. Proportion).

Cor. 3. *In a right cone, as the radius of the base* BC : *to the side* AB : : *so the area of the base* BD : *to the curve surface of the cone* ABD.

For it is $\frac{BC \times BKDB}{2}$: $\frac{AB \times BKDB}{2}$: : BC : AB.

Cor. 4. *The curve surfaces of right cones, are in the complicate ratio of the sides and diameters of the bases.*

For

For the equal triangles are in that ratio (Cor. 1. 11. II), and the diameters are as the circumferences (9. IV). FIG. 165.

Cor. 5. *The curve surface of a right cylinder, is to the curve surface of a right cone, on the same base; as the side of the cylinder, to half the side of the cone.*

PROP. VI.

The curve surface of the frustum of a right cone PD, *is equal to half the rectangle under the side* PB, *and the sum of the circumferences of the bases*, PE, BD. 166.

Produce BP, DE to A, and compleat the cone; then from A draw OI, FK exceeding near one another, then the small part of the curve surface OK, falls in with the small trapezoid OFKI, whose area is $\frac{\overline{OF + IK}}{2} \times OI$ (23. III). And as all the surface of the frustum may be divided into such trapezoids, therefore its surface is = sum of all the trapezoids = sum of all the $\frac{OF + IK}{2} \times OI = \frac{BKDB + POEP}{2} \times OI$.

Cor. *The curve surface of the frustum of a right cone, is equal to a circle, whose radius is a mean proportional between the side* PB, *and the sum of the radii of the bases*, BC + PH.

For let R = radius, C = $\frac{1}{2}$ circumference, of the circle equal to the surface of the cone ABD. And r = radius, c = $\frac{1}{2}$ circumference of the circle equal to the surface of the cone APE. And since R : C : : r : c (9. IV), let $\frac{C}{R} = \frac{c}{r} = n$, or C = R$n$, and $c = rn$. The triangles APH, ABC are similar, and BC : PH : : BA : PA (13. II), and BC — PH

 : PH

FIG. 166. $: PH :: PB : PA = \frac{PH \times PB}{BC - PH}$ (13. Proportion); but ſurface of the cone ABD $= RC = nRR = n \times AB \times BC$ (Cor. 2. 5) $= n \times \overline{AP + PB} \times BC = n \times \overline{\frac{PH \times PB}{BC - PH} + PB} \times BC = n \times \frac{PH \times PB + BC \times PB - PH \times PB}{BC - PH} \times BC = n \times \frac{BC \times PB}{BC - PH} \times BC.$

Alſo ſurface of the cone APE $= rc = nrr = n \times AP \times PH$ (Cor. 2. 5) $= n \times \frac{PH \times PB \times PH}{BC - PH}$. Therefore their difference, or the ſurface of the fruſtum is $n \times \frac{BC^2 \times PB}{BC - PH} - n \times \frac{PH^2 \times PB}{BC - PH} = n \times PB \times \frac{BC^2 - PH^2}{BC - PH} = n \times PB \times \overline{BC + PH}$ (12. I) $=$ the circle whoſe radius is $\sqrt{PB \times \overline{BC + PH}}$, and circumference $n \times \sqrt{PB \times \overline{BC + PH}}$.

PROP. VII.

167. *The ſurfaces of ſimilar ſolids* AD, PS, *are as the ſquares of their homologous ſides,* AB^2 *and* PQ^2.

Draw the diagonals AC, PR. Then ſince the bodies are reſolvable into ſimilar pyramids (Def. 12), which are contained under ſimilar plane figures (Def. 11). Let the planes incloſing them, be ABC, PQR, and AGC, PIR, &c. which being ſimilar, it is AB : PQ :: AC : PR :: AG : PI :: GE : IT, &c. (13. II); and ſince $AB^2 : PQ^2$ :: triangle ABC : PQR (18. II), and $AC^2 : PR^2$ :: ACG : PRI; and $AG^2 : PI^2$ :: trapezium AE : PT (20. III); and $GE^2 : IT^2$ :: GD : IS, &c. therefore $AB^2 : PQ^2$ :: ABC : PQR :: ACG : PRI :: AE : PT :: GD : IS &c. whence $AB^2 : PQ^2$ :: BG + AE + GD &c. : QI + PT + IS &c. (10. Proportion) :: ſurface of AD : ſurface of PS. Cor.

Cor. *Similar parts of the surfaces of similar solids, are as the squares of the homologous sides.* FIG. 167.

PROP. VIII.

A right triangular prism ABCFHE *is equal to an oblique one* APIGHD, *of the same length* AH, *contained within the same three parallel lines* EP, HA, FI, *or the planes passing through them.* 168.

For AH = PD = BE = IG = CF (1. III), whence PB = DE, IC = GF, and AP, AB, AC, AI being parallel to HD, HE, HF, HG (Cor. 3. 5. I), the solid angle A = solid angle H (20. V). For the same reason the solid angles at P, B, C, I, are respectively equal to those at D, E, F, G. And since the sides are all equal, each to each, therefore the two solids APBCI and HDEFG will exactly coincide, and be equal the one to the other (Ax. 8); and therefore the rectangled prism HEFCAB = the oblique one HDGIAP.

Cor. 1. *If a parallelopipedon* AB, *be cut by a plane passing through the diagonals of the opposite planes; it shall be cut into two equal parts.* 169.

For the triangle CGF = CBF, and DAE = DHE (1. III); and the length AG = length BH; therefore if the two prisms CFA, and CFH be laid so, that H may coincide with A, and EHB with DAG, their planes will concide, and each of them being oblique, is equal to a right one of the same length (8).

Cor. 2. *Hence any prismatic solid cut obliquely by parallel planes, is equal to the same cut off at right angles, and of the same length.*

For any such solid may be divided into triangular prisms, by planes passing through both ends of the solid.

FIG. 169. ſolid. And each triangular priſm cut obliquely, is equal to one of the ſame length, cut at right angles (8).

PROP. IX.

170. *If a parallelopipedon* AS, *be cut by a plane, paſſing through* O *the middle of the diameter* CQ; *the plane biſects it.*

Let the diagonals AD, BC cut each other in F; and RQ, PS, in I. Draw the axis FI, which cuts CQ in O, becauſe BCRQ is a parallelogram (2. and Cor. 3. III); and FO = OI. Let the plane EHOVX be parallel to ABDC. Then the parallelopip. AX = half AS. Let any plane GTOLN paſs through O. And let the ſolid be cut by the two planes ADSP, and CBQR, into four triangular priſms.

The two oppoſite ſolids OTGEH and OLNXV, are equal; for the ſides are parallel (11. V), and equal (Cor. 3. III). And therefore the ſolid angles, at the correſpondent points, are equal (20. V); therefore the ſolid EOG = XON. Therefore in the oppoſite priſms ACI, and BDI, the ſolids contained between the planes EVXH and GTLN, are equal. And it is proved the ſame way, that the ſolids, in the oppoſite priſms ABI, and DCI, contained between the planes EVXH and GLNT, are equal. And therefore ſince AX is half the parallelop. the plane GTNL cuts off half the parallelop. or divides it into two equal parts.

Cor. *The axis* FI, *and diagonal* CQ, *biſect each other.*

For they are both in the parallelogram BCRQ (Cor. 3. III).

PROP.

PROP. X. FIG.

Parallelopipedons upon the same base CDFI, *and between the same parallel planes*, CIFD *and* BHVOLA, *are equal.* 171.

The triangles LAI and KEF are equal and similar (6. II); and the prism KEFDQH = LAICBG; subtract the common solid E*r*LQ*s*G, and the solid AI*r*EBC*s*Q = L*r*FKG*s*DH; add the prism I*r*FC*s*D, and the solid paral. CDFIQEAB = CDFIHKLG upon the same base ID.

Again, the triangles FVK and DMH are similar and equal (6. II), and the prism FVKIOL = DMHCLG; subtract the common prism M*t*KP*n*L, and the solid FVM*t*IOP*n* = D*t*KHC*n*LG; add the prism DF*t*CI*n*, and the solid par. FVMDIOPC = DFKHCILG = CDFIQEAB.

PROP. XI.

Parallelopipedons of equal bases and hights are equal. 172.

Let the parallelogram AGIC be the base of the parallelopiped. Draw BH, DF parallel to AG, AC. The solid pip. upon the base AGI = that on the base ACI (Cor. 1. 8); and solids on ABE and EFI, are = those on ADE, and EHI (ibid.). Take the two last from the first, and there remains the solid on DH = solid on BF. But parallelogram DH = BF (4. III). Therefore solid pips. on equal bases and hights are equal, when the angle at E is the same. Moreover, the pip. on the base BCEF is equal to that on the base EOPF, and the same hight (Cor. 2. 8); reckoning OP or EF the length of the solid. Whence the parallelopip. on the base DH, is equal to the pip. on the base EP, and hight the same.

PROP.

FIG.

PROP. XII.

173. *Parallelopipedons of the ſame hight are in proportion as their baſes.*

Let BN be the baſe of a parlopip. divide the baſe into any number of equal parts at D, E, F, G, &c. and draw planes || to ABC; then the parlopips. ſtanding upon CD, DE, EF, &c. will be all equal (11); whence the pip. AK is as multiple of AD, as the baſe BK is of the baſe BD, alſo the pip. LN is as multiple of AD, as the baſe ON is of BD. Whence it will be as pip. AK : pip. LN : : baſe BK : baſe ON (Def. 4. Proportion). Moreover, let the baſe PQ be = ON, and hight QR = AB, then the pip. PR = LN (11), whence pip. AK : pip. PR : : baſe BK : baſe PQ.

Cor. 1. *Parallelopipedons of equal baſes are as their hights.*

For in rectangled ones, any ſide or face may be taken for the baſe; and rightangled ones are equal to oblique ones, between the ſame parallel planes (10).

Cor. 2. *Parallelopipedons are to one another, in the complicate ratio of their baſes and hights.*

PROP. XIII.

175. *If two parallelopipedons,* AD, FI, *be equal; their baſes and hights are reciprocally proportional;* AC : FH : : HI : CD.

Suppoſe the ſides CD, HI perpendicular to the baſes, and make HM = CD. Then baſe AC : baſe FH : : ſolid AD or FI : ſolid FM (12) : : HI : HM or CD (Cor. 1, 12). And if the pips. be oblique, inſtead of ſuppoſing CD, HI to be the ſides,

let

let them be the hights, and then oblique pips. being equal to upright ones (10); the proportion continues the same. FIG. 175.

Cor. *If the bases and hights of two parallelopipedons be reciprocally proportional, they are equal.*

For since base AC : base FH : : HI : CD (hyp.), therefore AC × CD = FH × HI (12. Proportion), and solid BE : solid FI : : AC × CD : FH × HI (Cor. 2. 12). Therefore solid BE = solid FI (Ax. 7. Proportion).

PROP. XIV.

All prisms whatsoever, ABD, PSR, *of equal bases and hights, are equal.* 174.

For any polygonal base BD may be divided into triangles, by diagonal lines; and the polygonal prism may likewise be divided into triangular prisms, by planes passing through these diagonals; each of which triangular prisms is equal to half a parallelopipedon standing on double the base (9); and as all these triangular prisms make up the polygonal prism, this prism must be equal to a parallelopip. of the same base and hight, and that equal to the prism PRS of an equal base and hight (Cor. 1. 12).

Cor. 1. *Prisms of equal bases are as their hights; and of equal hights, are as their bases.*

For they may be divided into triangular prisms, which are half of parlopips. on double the base, and these pips. are as their hights, when the base is the same; or as the bases, when the hight is the same. (Cor. 2. 12).

Cor. 2. *All prisms are to one another in the complicate ratio of their bases and hights.*

Cor.

FIG. 174. Cor. 3. *Bodies of equal surfaces may be very different in solidity. And equal solids may have surfaces vastly different.*

Cor. 4. *In equal prisms, the bases and hights, are reciprocally proportional; and the contrary.*

PROP. XV.

The solidity of any prism is equal to the product of the base and hight.

For a prism is equal to a right-angled parallelopip. of the same base and hight; and that is equal to the product of its base and hight; or (which is the same) it is equal to the solid space contained under the planes of the upright parallelopipedon (Def. 5).

PROP. XVI.

176. *Equiangular parallelopipedons* AB, CD, *are in the complicate ratio of their homologous sides*, FG, GI, GB, *and* OE, EH, ED.

Let FP, OK be ⊥ upon the bases IB, HD. Then by reason of the equal angles at G and E, the triangles GFP, EOK will be similar; and FP : OK : : FG : OE (13. II). The parallelograms IB and HD being equiangular at G and E, are to one another as IG × GB, to HE × ED (10. III). The parlepip. AB : CD : : base IB × FP : base HD × OK (Cor. 2. 12) : : IG × GB × FP : HE × ED × OK : : GI × GB × GF : HE × ED × EO (7. Proportion).

PROP. XVII.

177. *Pyramids upon the same base, and of equal attitudes, are equal:* ACF = HCF.

Draw

Draw the plane AH, through the tops of the pyramids, which will be parallel to CF. Alſo through any points of the pyramids, draw the plane BE, alſo parallel to CF; then by ſimilar triangles, CF : BD : : AC : AB (13. II) : : HC : HL (12. II) : : CF : LE (13. II); therefore BD = LE. And by the ſame reaſoning, BO = LI, and DO = EI. Whence the ſection BOD = LIE (8. II). Therefore if another plane NP be drawn very near, and parallel to BE, the ſegments of the pyramids, ND, PL, comprehended between theſe planes, will be equal (14). And therefore if never ſo many ſuch planes be drawn, the parts intercepted will always be equal. Therefore the ſum of all the parts of one pyramid, will be equal to the ſum of all the parts of the other; or the pyramid ACGF = pyramid HCGF (Ax. 2). FIG. 177.

Cor. 1. *If a pyramid is cut by a plane parallel to the baſe, the ſection will be ſimilar to the baſe.*

For by ſimilar triangles, it is AC : AB : : CG : BO : : GF : OD : : CF : BD.

Cor. 2. *If a cone be cut by a plane parallel to the baſe; the ſection will be a circle.*

For a cone may be conſidered as a pyramid of an infinite number of ſides.

PROP. XVIII.

Every priſm is three times the pyramid of the ſame baſe and hight. 178.

Let AFC be a triangular priſm, draw AC, CF, FD, the diagonals of the three parallelograms. The triangle ACB = ACD (1. III); therefore pyramid ACBF = ACDF, their vertexes being in F (17); likewiſe triangle DFA = DFE (1. III), and pyramid DFAC = DFEC, their vertexes being

FIG. 178. ing in C (17). But ACDF and DFAC are one and the ſame pyramid. Therefore the three pyramids, that make up the priſm, are equal to one another, ACBF = ACDF = DFEC; and each of them is $\frac{1}{3}$ the priſm.

And ſince any polygonal priſm may be reſolved into triangular ones; and the pyramid, upon the ſame baſe, into triangular pyramids. Then all the triangular priſms will be triple to all the triangular pyramids; and conſequently the whole priſm triple to the whole pyramid.

Cor. 1. *Pyramids of the ſame hight, are to one another as their baſes.*

For priſms, which are triple of them, are in that ratio (Cor. 1. 14). Whence,

Cor. 2. *Pyramids of the ſame or equal baſes are as the hights.*

Cor. 3. *Pyramids are to one another in the complicate ratio of their baſes and hights.*

Cor. 4. *Pyramids of equal baſes and hights are equal.*

Cor. 5. *In equal pyramids the baſes and hights are reciprocally proportional; and the contrary.*

For priſms are in that ratio (14. and Cor.).

PROP. XIX.

Cylinders of equal baſes and hights are equal.

For cylinders are nothing but priſms, whoſe baſes are polygons of an infinite number of ſides. And theſe priſms are equal (14).

Cor. 1. *Cylinders of equal baſes are as the hights.*

Cor. 2. *Cylinders of equal hights are as the baſes.*

Cor.

Cor. 3. *Cylinders are to one another in the complicate ratio of their bases and hights.* FIG. 178.

Cor. 4. *In equal cylinders, the bases and hights are reciprocally proportional: and the contrary.*

All this follows from Prop. 13, 14, and Corol.

PROP. XX.

Every cone is the third part of a cylinder of the same base and hight.

For cones and cylinders may be considered as pyramids, and prisms, whose bases are regular polygons of an infinite number of sides. And consequently the cone $= \frac{1}{3}$ the cylinder (18).

Cor. 1. *Cones of equal bases, are as their hights.*

Cor. 2. *Cones of equal altitudes, are as the bases.*

Cor. 3. *Cones are to one another in the complicate ratio of the bases and hights.*

Cor. 4. *In equal cones, the bases and hights are reciprocally proportional.*

All these things appear by Prop. 13 and 14, and 19. For the cylinders are in that ratio, and the cone is $\frac{1}{3}$ the cylinder.

PROP. XXI.

The frustum of a pyramid or cone BG, *is equal to the third part of a parallelopipedon, of the same hight, and its base equal to the sum of the bases of the frustum* BOD + EFG, *together with a mean proportional between these bases.* 179.

Draw EB, GD to meet in A, the top of the whole solid, and let ACP be ⊥ to the base. Draw the diameters BD, EG; then the two bases BOD,

FIG. EFG will be similar (Cor. 1, 2. 17). Whence,
179. base BOD : base EFG :: $BD^2 : EG^2$ (20. III). Therefore suppose $\frac{\text{base BOD}}{BD^2} = \frac{\text{base EFG}}{EG^2} = n$, or base BOD $= n \times BD^2$, and base EFG $= n \times EG^2$. By similar triangles, EG : BD :: (AE : AB ::) AP : AC (13. II), and EG — BD : BD :: CP : $AC = \frac{BD \times CP}{EG - BD}$. Then the whole pyramid or cone $=$ base EFG $\times \frac{1}{3}$AP (18, 20) $= \frac{n \times EG^2}{3} \times \overline{CP + AC} = \frac{n \times EG^2}{3} \times CP + \frac{n \times EG^2}{3} \times \frac{BD \times CP}{EG - BD}$ $= \frac{n \times EG^3 \times CP - n \times EG^2 \times BD \times CP + n \times EG^2 \times BD \times CP}{3 \times \overline{EG - BD}}$ $= \frac{n \times EG^3 \times CP}{3 \times \overline{EG - BD}}$. And the top part ABD $= \frac{\text{base BOD}}{3} \times AC$ (18, 20) $= \frac{n \times BD^3 \times CP}{3 \times \overline{EG - BD}}$; this taken from the whole, leaves $\frac{n \times CP}{3} \times \frac{EG^3 - BD^3}{EG - BD}$ for the frustum $= \frac{CP}{3} \times \overline{n \times EG^2 + n \times EG \times BD + n \times BD^2}$, because $\overline{EG^2 + EG \times BD + BD^2} \times \overline{EG - BD} = EG^3 - BD^3$ (Cor. 1. 8. I), and $n \times EG^2 =$ base EFG, $n \times BD^2 =$ base BOD, and $n \times EG \times BD$ is a mean between them (Cor. 2. 12. Proportion).

Cor. *If* $n = \frac{\textit{base}\ \text{EFG}}{EG^2}$, *the frustum* $= \frac{n \times CP}{3} \times \frac{EG^3 - BD^3}{EG - BD}$.

PROP. XXII.

180. *In similar solids,* AD, PS, *the homologous sides are*
181. *proportional*; AB : AF :: PQ : PV.

Through the diagonals AC, FG, GD, and PR, VI, IS, let planes be drawn to divide the solids into pyramids. Then since these pyramids are similar (Def. 12), and their planes similar figures (Def. 11); therefore if ABC, PQR, and ACG, PRI, and AGF, PIV, &c. be similar planes belonging to the similar pyramids; it will be AB : PQ (: : AC : PR : : AG : PI) : : AF : PV. Also AF : PV : : (FG : VI : :) FE : VT, &c. FIG. 180. 181.

Cor. *The like planes or surfaces, which inclose similar solids, are proportional.*

For since AB : PQ : : AF : PV; $AB^2 : PQ^2$: : $AF^2 : PV^2$ (Cor. 3. 18. Proportion); that is, ABCG : PQRI : : AGEF : PITV (20. III).

PROP. XXIII.

Similar triangular pyramids ABCD, PQRS *are as the cubes of their homologous sides*, AB^3 *and* PQ^3. 182. 183.

Suppose CE, BF drawn parallel to AD, and RT, QV, ∥ to PS; and the planes DFE, SVT, ∥ to ABC, and PQR; and so the prisms AF, and PV, compleated.

Then since the pyramid ABCD = $\frac{1}{3}$ prism, AF; and pyramid PQRS = $\frac{1}{3}$ prism PV; therefore pyramid ABCD : pyramid PQRS : : prism AF : prism PV (5. Proportion) : : AB × AC × AD : PQ × PR × PS (16).

But AB : PQ : : AB : PQ,
AB : PQ : : AC : PR (22),
AB : PQ : : AD : PS (22).

Therefore $AB^3 : PQ^3$: : AB × AC × AD : PQ × PR × PS (18. Proportion) : : pyramid ABCD : pyramid PQRS.

Cor. *Any similar pyramids are as the cubes of the homologous sides.*

FIG. For they may be divided into ſimilar triangular
182. pyramids, all which are in that proportion, and
183. their ſums in the ſame proportion (10. Proportion).

PROP. XXIV.

180. *All ſimilar ſolids*, AD, PS, *are to one another,*
181. *as the cubes of their homologous ſides*, AB, *and* PQ.

Let the planes AC, PQ, and FG, VI, and GD, IS, &c. divide the bodies into ſimilar pyramids. Then ſince AB : PQ : : AG : PI : : EG : TI, &c. (22). Therefore

$AB^3 : PQ^3$: : pyr. ABC : pyr. PQR (23),
and $AB^3 : PQ^3 : : AG^3 : PI^3$: : pyr. AGC : pyr. PIR : : pyr. AGF : pyr. PIV.
and $AB^3 : PQ^3 : : EG^3 : TI^3$: : pyr. FGE : pyr. VIT : : pyr. EGD : pyr. TIS, &c.

Therefore

$AB^3 : PQ^3$: : pyr. ABC + AGC + AGF + FGE + EGD, &c. : pyr. PQR + PIR + PIV + VIT + TIS, &c. : : ſolid AD : ſolid PS.

Cor. *If four lines* A, B, C, D *be in continual proportion; then as the firſt* A *to the fourth* D; *ſo any ſolid deſcribed on the firſt* A, *to a ſimilar one, on the ſecond* B.

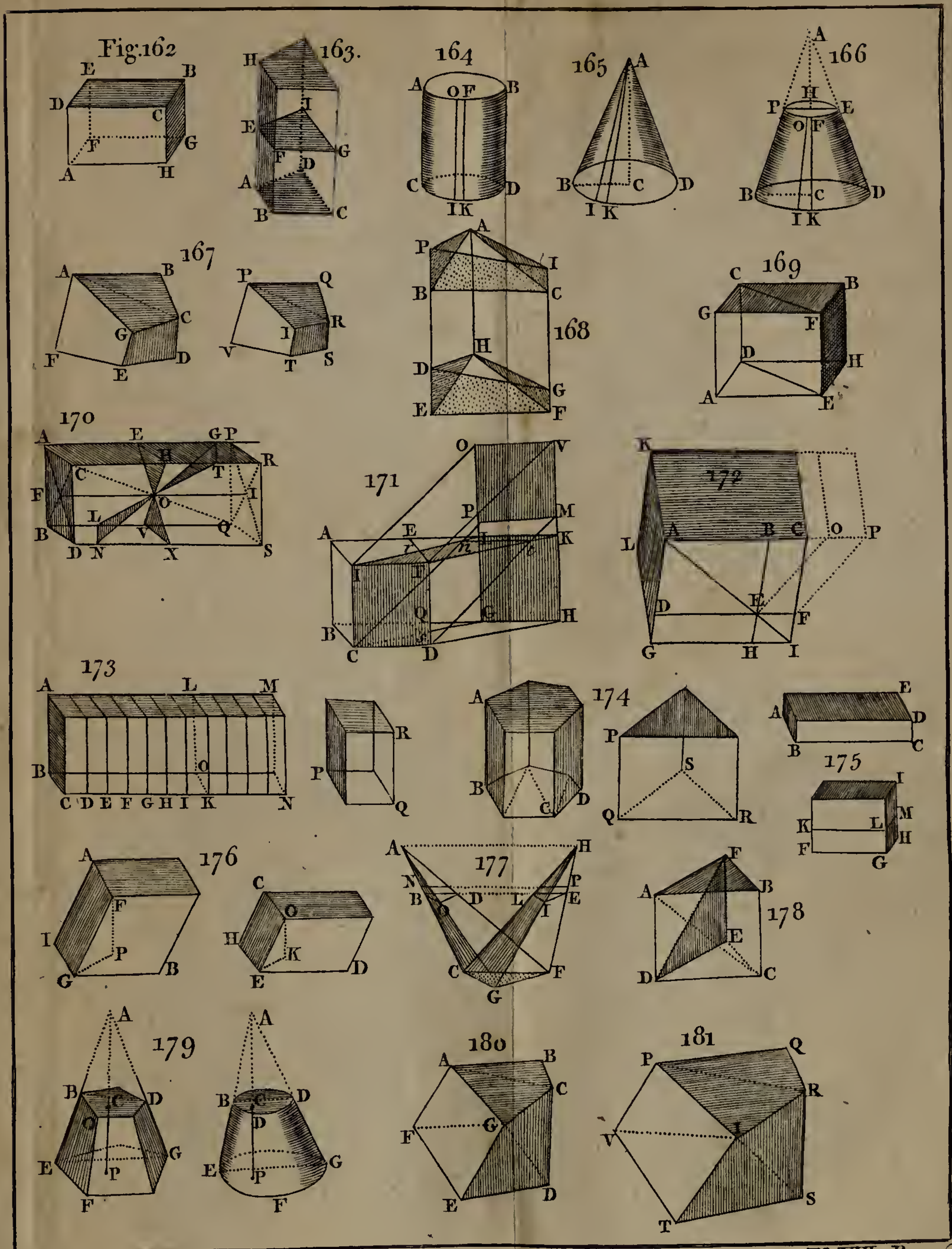
For A : D : : $A^3 : B^3$ (23. Proportion) : : ſolid upon A : ſolid upon B (24).

PROP. XXV.

184. *If four lines be proportional*, AB : CD : : GH : LM; *ſimilar ſolids, alike deſcribed, upon two and two, ſhall alſo be proportional:* ABE : CDF : : GHK : LMN.

And if four figures be proportional, and two and two ſimilar; their homologous ſides ſhall be proportional.

For



Pl. VIII. *Pa. 116.*

For ſince AB : CD : : GH : LM (hyp.), FIG. 184.

therefore $AB^3 : CD^3 :: GH^3 : LM^3$ (Cor. 3. 18. Proportion),

whence ABE : CDF : : GHK : LMN (24).

Again, if the ſolids be ſimilar,

and ABE : CDF : : GHK : LMN (hyp.),

then $AB^3 : CD^3 :: GH^3 : LM^3$ (24),

whence AB : CD : : GH : LM (Cor. 3. 18. Proportion).

PROP. XXVI.

None other but three ſorts of regular plane figures, joined together, can make a ſolid angle; and theſe are, 3, 4, or 5 triangles, 3 ſquares, and 3 pentagons:

And therefore there can only be five regular bodies, the pyramid, cube, octaedron, dodecaedron, and icoſaedron.

Three plane angles at leaſt, are required to make a ſolid angle. One angle of the triangle = $\frac{2}{3}$ of a right angle (2. II), therefore 3 of them put together make two right angles. Alſo 4 of them make $2\frac{2}{3}$ right angles. And 5 make $3\frac{1}{3}$ right angles; all which are leſs than 4 right angles. But 6 of them make 4 right angles, and therefore cannot make a ſolid angle (17. V).

Again, one angle of the ſquare is a right angle, and 3 of them make 3 right angles. But 4 make 4 right angles, and therefore can make no ſolid angle (17. V).

Alſo one angle of the pentagon is $1\frac{1}{5}$ right angle (17. III). And 3 angles make $3\frac{3}{5}$. But 4 of them make $4\frac{4}{5}$, which exceeds 4 right angles.

Laſtly, one angle of the hexagon is $\frac{4}{3}$ of a right angle, therefore 3 angles make 4 right angles; but no ſolid angle. And the angle of a heptagon, octagon, &c. being greater; 3 of them will exceed 4 right angles; and conſequently, there can be no

FIG. more than 3 triangles, 1 ſquare, and 1 pentagon,
184. to conſtitute a ſolid angle.

Hence there can only be 5 regular bodies, to anſwer the 5 combinations of triangles, ſquares, and pentagons. Three faces of the triangle make the *pyramid*; 4 make the *octaedron*; and 5 make the *icoſihedron*; alſo 6 faces of the ſquare make the *cube*; and 12 faces of the pentagon, make the *dodecaedron.*

SCHOLIUM.

185. In order to get a clear idea of the five regular
186. bodies, you may cut out all their faces in paſte-
187. board, as repreſented in the figures, and fold them
188. up, ſo that the creaſes may be in the black lines;
189. and their edges being put cloſe together, you'll have the figure of theſe bodies. Fig. 185 is the pyramid, 186 the cube, 187 the octaedron, 188 the dodecaedron, and 189 the icoſihedron.

PROP. XXVII.

No other but only one ſort, of the five regular bodies, joined at their angles, can compleatly fill a ſolid ſpace; and that is eight cubes.

To demonſtrate this, we muſt obſerve that among other properties, this is abſolutely neceſſary, that the inclination of two adjoining planes in the body, be ſuch; that being taken a certain number of times, they will compleatly make up four right angles. For when the bodies are put together, the faces of every two adjoining bodies muſt coincide; and one edge or ſide of all the bodies muſt coincide with the ſide of the firſt; which will be as an axis, round which theſe bodies are placed; and therefore they muſt compleatly fill up the ſpace quite round, which is four right angles. And the

the angle of each (that is, the inclination of two FIG. adjoining planes), muſt be a certain part of 4 right angles. Therefore what we have to do, is to compute the inclination of their planes, and alſo to enquire what inclination is requiſite in the ſeveral bodies, to have this effect.

1. To begin with pyramids. It is plain, the baſe of the ſolid, being an equilateral triangle, the angle at any point is $\frac{2}{3}$ of a right angle; but the inclination of the planes is greater; for it is contained by two perpendiculars let fall on the common ſection of two planes, which perpendiculars are leſs than the ſides of the triangle (Cor. 4. 21. II); and ſtanding on the ſame baſe, muſt contain a greater angle (Cor. 2. 5. II). To 190. find the inclination of the planes; let CPH, CPA, and CDH be three of the equilateral triangles conſtituting a pyramid. Draw AG, DI $\perp$ to CP, CH. Let the plane CHP be fixt, whilſt the planes CAP, CDH, are raiſed up, (moving about the fixt lines CP, CH,) till the points A and D meet ſomewhere. It is plain a perpendicular dropt from A (elevated on high), upon the plane CPH, will always be ſomewhere in the line AG. And a like perpendicular from D will be ſomewhere in the line DI. Therefore when A and D meet, the perp. will be at the interſection O, in the middle of the triangle; and GO $= \frac{1}{3}$GH (Cor. 31. II) $= \frac{1}{3}$GA. Therefore, if you make the ſeparate right-angled triangle GAO, ſo that the hyp. GA may be treble the baſe GO, the $\angle$AGO is the angle of the pyramid, (that is, of its planes CAP, CHP), which was required. Now if EG be $\perp$ to GK, alſo if GBK be an equilateral triangle, then the baſe GF, will be half the hypothenuſe GB (Cor. 3. 3. II), and $\angle$BGK $= \frac{2}{3}$ a right angle (2. II). Then its plain, 4 times $\angle$AGK will be leſs than 4 right angles, becauſe 4 times EGK make but 4 right angles;

FIG. 190. gles; therefore more than 4 times AGK is required to compleat 4 right angles. Likewise, since 6 times BGK make 4 right angles, 6 times AGK will be too much; and of consequence we must either have 5 times AGK, to make 4 right angles, or

134. nothing. Then to find whether that will answer exactly or not; draw the diagonal EC of the pentagon, and OLD ⊥ to it; then 5 times the angle EOL = 4 right angles. But DL $= \frac{5+\sqrt{5}}{4}$R (Cor. 3. 44. IV), and OL $= R - R \times \frac{5-\sqrt{5}}{4} =$

190. 134. $\frac{\sqrt{5}-1}{4} \times R$. But GO (fig. 190) $= \frac{1}{3}$ the hypothenuse AG or R, and $\frac{1}{3}$ is greater than $\frac{\sqrt{5}-1}{4}$, that is, GO is greater than OL, and consequently the angle AGO is lesser than EOL, which it should be equal to; therefore 5 times AGO falls short of 4 right angles; whence it is clear, that no combination of regular pyramids can compleatly fill all space.

2. And it is as clear that 4 cubes set together will make up 4 right angles, each cube containing one. And therefore 8 cubes, joined at their angular points, will quite fill all space on all sides.

191. 3. Next for the octaedron. As half the octaedron ABE stands on a square base BCED, the angles at the base, as BCE, are right, and then 4 of these would be 4 right ones; but the inclination of the planes ACB, ACE, are greater than right angles (for the same reason as in the pyramid), being made by a plane ⊥ to their common section AC; therefore 4 of these angles will be too much, and consequently 3 or none of these angles of inclination must be equal to 4 right angles; or, which is the same thing, 6 halfs of the ∠ of inclination must be = 4 right angles. Now to try this, draw AG ⊥ to BC,

BC, and AO ⊥ to the base BE, also draw GO. Then FIG. hyp. AG $= \frac{AB}{2}\sqrt{3}$ (Cor. 39. II), and base GO $=$ 191.
$\frac{1}{2}$BD $= \frac{1}{2}$AB. Therefore AG : GO :: $\frac{AB}{2}\sqrt{3}$: $\frac{AB}{2}$ :: $\sqrt{3}$: 1 :: 3 : $\sqrt{3}$ :: 1 : $\frac{\sqrt{3}}{3}$ :: AG : $\frac{AG}{3}\sqrt{3}$; then GO $= \frac{AG}{3}\sqrt{3}$. And as ∠AGO = half the angle of inclination, 6 of these must make up 4 right angles. And therefore ∠AGO must be = ∠BDE (fig. 131), if this succeed. For 6 of these make up 4 right angles. But in this case, DF $= \frac{1}{2}$DB, whence if DB (fig. 131) = AG (fig. 191), then GO $= \frac{DB}{3}\sqrt{3}$. But $\frac{1}{3}\sqrt{3}$ is greater than $\frac{1}{2}$ (as is easily known by squaring them); that is, GO is greater than DF, and consequently ∠AGO is less than BDF. Therefore 6 of these, or 3 whole angles of inclination, fall short of 4 right angles. So these bodies cannot entirely fill all space.

4. Next comes the dodecaedron. As the angle of inclination of the planes of this body exceeds a right angle; therefore 4 such angles will exceed 4 right angles; therefore only three of these bodies can be laid together; in which case the angle of inclination must be just $1\frac{1}{3}$ right angle For $3 \times 1\frac{1}{3} = 4$. If the ∠ be less, the third body will leave a vacuity; if greater, it cannot come in. Let 192.
BPC, PCH, DGH, be 3 pentagons joining upon one another. Draw AG, DI ⊥ to PC, HC, continued. Then let the plane PCH, be fixt, whilst ABP, DEH, are raised up, and moved round the lines PC, HC, till the points A, D, meet. It is evident a perpend. dropt from A upon the plane PCH, will always fall on the line AG. And a like perpend. from D, will fall upon DI. And when A and D meet, it will fall on the intersection O.

Let

FIG. 192. Let R ſtand for a right angle. Then ſince CE is ∥ to HN (Cor. 2. 43. IV), ∠ECH + CHN = 2R (Cor. 2. 4. I) = ECH + PCH, therefore PCE is a right line (1. I). For the ſame reaſon BCH is a right line. Since ∠DCH = $\frac{6}{5}$R (17. III), DCE = $\frac{2}{5}$R, DCP = $\frac{8}{5}$R, take away ACP = $\frac{6}{5}$R, then ACD = $\frac{2}{5}$R. In the iſoſceles triangle ACD, COF biſects the ∠C and baſe AD (Cor. 3. 3. II), and ∠ACF = $\frac{1}{5}$R = DCF, and CDA = $\frac{4}{5}$R; and ſince CDE = $\frac{6}{5}$R, therefore CDA + CDE = 2R, and EDA is a ſtraight line (1. I). In the right-angled triangle ACF, ∠ACF or ACO = $\frac{1}{5}$R; and in the right-angled triangle ACG, ſince ACE = ACD + DCE = ACG = $\frac{4}{5}$R, CAG = $\frac{1}{5}$R = ACO, or CAO = ACO, and AO = OC (Cor. 1. 3. II). Therefore OG is leſs than OC or OA (5. II), and OG is leſs than half of AG. Make a right angle triangle ſeparately, as AGO, where the hypothenuſe is AG, and baſe OG, of a due length, and AGR is one of the angles of the dodecaedron. Where the ∠AGZ or GAO ought to be $\frac{1}{3}$R, that 3 dodecaedrons laid together may fill up 4 right angles. Now to ſee how this agrees, we find (in fig. 128), that EF = $\frac{1}{2}$DE, or DF = $\frac{1}{2}$DB (Cor. 3. 41. IV), and ∠ABF or BAC = $\frac{2}{3}$R = BDF (Cor. 1. 12. IV), and conſequently DBF = $\frac{1}{3}$R (Cor. 2. 2. II). Therefore if you make the baſe GQ = $\frac{1}{2}$ the hypothenuſe GM, then the ∠GMQ or MGZ is = $\frac{1}{3}$R. Therefore, ſince GO is leſs than $\frac{1}{2}$GA, the ∠AGZ is leſs than $\frac{1}{3}$R, and MGR leſs than 1$\frac{1}{3}$R, to which it ſhould have been equal; and conſequently 3 times MGR falls ſhort of 4 right angles: therefore the dodecaedrons cannot fill a ſolid ſpace.

This might be otherwiſe ſolved, by ſuppoſing one of its ſolid angles to ſtand upon an equilateral triangle, whoſe ſide is the diagonal of the pentagon.

5. Laſtly,

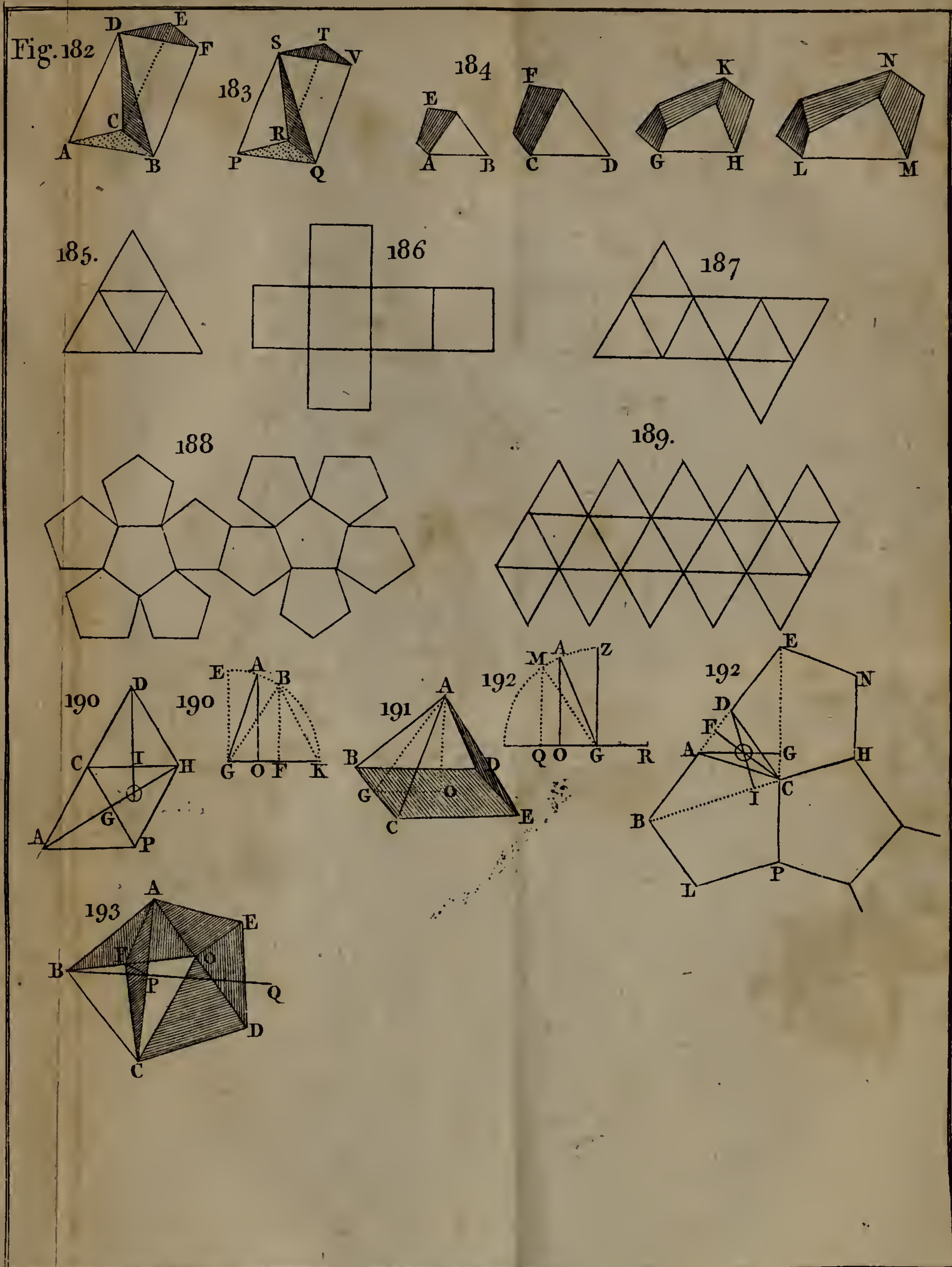
5. Laftly, the icofaedron has 5 triangles ftanding upon a pentagonal bafe ABCDE. Draw the diagonal AC of the pentagon, and BQ the diameter of the circumfcribing circle. And let the plane AFC be drawn at right angles to BO, the common fection of the two faces of the folid ABO, CBO. Draw FP, which will be $\perp$ to AC. Then we are to find the quantity of the $\angle$ AFC, the inclination of the planes, or rather, of its half AFP. Call $\frac{1}{2}$BQ, the radius of the circle, R; then AP^2 (Cor. 2. 44. IV) $= \frac{5+\sqrt{5}}{8}RR$. Alfo $AB^2 = RR \times \frac{5-\sqrt{5}}{2}$ (44. IV), and $AF^2 = \frac{3}{4}AB^2$ (Cor. 39. II) $= \frac{3}{4}RR \times \frac{5-\sqrt{5}}{2}$. Therefore $AF^2 : AP^2 :: \frac{3}{4}RR \times \frac{5-\sqrt{5}}{2} : \frac{5+\sqrt{5}}{8}RR :: 15 - 3\sqrt{5} : \overline{5+\sqrt{5}}$. And $AF^2 : AF^2 - AP^2$ or $FP^2 :: \overline{3 \times 5 - \sqrt{5}} : 10 - 4\sqrt{5} :: 3 : \frac{10-4\sqrt{5}}{5-\sqrt{5}} :: 3 : \frac{\overline{10-4\sqrt{5}} \times \overline{5+\sqrt{5}}}{\overline{5-5\sqrt{5}} \times \overline{5+\sqrt{5}}} :: 3 : \frac{50+10\sqrt{5}-20\sqrt{5}-20}{25-5=20}$ (Cor. 1. 8. I) $:: 3 : \frac{30-10\sqrt{5}}{20} :: 3 : \frac{2-\sqrt{5}}{2} :: 1 : \frac{3-\sqrt{5}}{6} :: 1 : .12732 :: AF^2 : .12732AF^2$. And by extracting the root, it is $AF : FP :: AF : .3568 \times AF = FP$. Now if three icofaedrons laid together can fill up the whole fpace, then three times the angle AFC, or fix times the $\angle$ AFP, muft make four right angles; and in that cafe AFP muft be $\frac{2}{3}$ of a right angle. But (fig. 128) the fide DF muft be half the hypothenufe DB, when the $\angle$ between them BDF is $\frac{2}{3}$ of a right angle (Cor. 3. 41. IV): for $\angle$ BDF = BAC in the equilateral triangle BAC (Cor. 1. 12. IV) $= \frac{2}{3}$ of a right angle (2, II). But here the fide FP is lefs than half AF or $.5 \times AF$; therefore the $\angle$ FAP will be lefs, and AFP greater, than it fhould be; that is,

FIG. 193.

FIG. 193. is, AFP is more than $\frac{2}{3}$ of a right angle; and 6 times AFP, more than 4 right angles; and therefore 3 icosihedrons cannot find room.

Thus I have demonstrated from pure geometrical principles, that no combination of regular bodies of the same sort (except cubes), can adequately fill up all the space round about. The calculations of all these cases are extremely easy, by working with the rules of trigonometry; but that was not my business here.

BOOK



Pl.IX.P

BOOK VII.

Of the ſphere, and its inſcribed and circumſcribed bodies.

DEFINITIONS.

1. A *Sphere or globe*, is a ſolid made by a ſemicircle ABD, moving round about its diameter AD, which remains fixt; and is called the *axis* of the ſphere; and the point A, the *vertex*. FIG. 194.

2. The *center* of the ſphere is the center C of the ſemicircle ABD.

3. The *radius* of the ſphere, is a line drawn from the center to the ſurface of the ſphere.

Cor. *All the radius of a ſphere are equal to one another.*

4. The *diameter* of a ſphere, is a right line drawn from one ſide to the other, through the center.

5. A *ſector* of a ſphere, CFDG, is a part of the ſphere made by the circular ſector FCD, moving round the radius CD. 195

6. *Segment* of a ſphere, is a part of a ſphere, as FIGD, cut off by a plane FIG. If the plane paſs through the center, that ſegment is a *hemiſphere*.

7. A *zone*, is a part of a ſphere intercepted between two parallel planes.

8. The *middle zone*, is the part between two parallel planes which are equally diſtant from the center.

9. A

FIG. 9. A solid is said to be *inscribed* in a sphere, or a sphere *circumscribed* about a solid; when all the angles of the solid touch the surface of the sphere.

201 10. A sphere is said to be *inscribed* in a solid, or a solid *circumscribed* about a sphere; when the sphere touches all the planes of the solid.

PROP. I.

196. *If a sphere be cut by a plane* FOG; *the section will be a circle.*

Let the two planes CFDG and COD be ⊥ to the cuting plane FOG; then the common section CI is ⊥ to the plane FOG (15. V). Draw the line FIG. Then in the triangles CFI, COI, CGI, the sides CF, CO, CG are equal (Cor. Def. 3), and CI common, and the angles at I right; therefore IF = IO = IG (9. II). Therefore FDG is a circle whose center is I (Cor. Def. 3. IV).

PROP. II.

197 *If a sphere* ABDI *touch a plane* HGL; *a right line* CD, *drawn from the center to the point of contact* D; *is perpendicular to the said plane.*

Let the planes ADB, ADF, cut the touching plane in the lines DH, DG. Then since HD, GD, touch the circles BD, FD, (whose center is C,) in D, therefore CD is ⊥ to HD, GD (Cor. 2, 10. IV); and therefore CD is ⊥ to the plane HGL (4. V).

PROP. III.

198. *The surface of a sphere is equal to the curve surface of its circumscribing cylinder.*

Let BAP be a hemisphere, and BHOP a cylinder on the same base, BTP, and of the same altitude.

tude. Take IL, an extremely ſmall part of the quadrant BLA, and through L and I, ſuppoſe two planes MLEVQ, and NIFSR to be drawn ⊥ to AC. Through L and I draw the line ILD, and through S and V the line SVG. Then becauſe IL and VS are extremely ſmall; the right lines and arches LI, VS nearly concide. And if the figure DISG be turned about the radius AC, it will generate the fruſtum of a cone; and the ſmall parts of its ſurface ILVS will concide with the portion of the ſpherical ſurface, and be equal thereto (Ax. 8). But the ſurface of the fruſtum ILVS is = IL × half the ſum of the circumferences whoſe diameters are LV and IS (6. VI), that is = IL × circumference of LV or IS, they being nearly equal. Let C = circumference whoſe radius is BC, and *c* = circumference whoſe radius is LE or IF, then the ſurface ILVS = *c* × IL, and the cylindric ſurface NMQR = C × MN (4. VI). But the triangles ILK, and LCE are ſimilar; for ∠ILC (Cor. 2. 10. IV) = KLE = a right angle; take away KLC, then ∠ILK = CLE; alſo ∠IKL = LEC = a right angle. Therefore LC or BC : LE :: LI : LK (13. II). But C : *c* :: BC or ME : LE (Cor. 9. IV) :: LI : LK or MN. Whence C × MN = *c* × IL (12. Proportion); that is, the cylindric ſurface NMQR = ſpherical ſurface ILVS. Therefore if more parallel planes, as MLVQ, be drawn, exceeding near to one another, the ſmall parts of the cylindric ſurface will be equal to the correſpondent parts of the ſpherical ſurface, and therefore the ſum of all the parts of the cylindric ſurface, equal to the ſum of all the parts of the ſpherical ſuface (Ax. 2); that is, the ſurface of the hemiſphere is equal to the ſurface of the cylinder BO, and the ſurface of the whole ſphere = ſurface of its circumſcribing cylinder.

FIG. 19 &

Cor.

FIG. 198. Cor. 1. *If the sphere and its circumscribing cylinder be cut by two planes parallel to the base, the intercepted parts of the surfaces of the sphere and cylinder, will be equal.*

For surface MR = surface LS, and all the MR = all the LS.

Cor. 2. *The surface of the hemisphere* BAP, *is double the base* BTP.

For the surface of the cylinder = C × AC (4. VI); and the area of the base = $\frac{C \times BC}{2}$, or $\frac{C \times AC}{2}$ (34. IV).

Cor. 3. *The surface of the whole sphere is equal to four great circles of the same sphere; or to the rectangle of the circumference and diameter.*

Cor. 4. *The areas of spherical surfaces cut off by parallel planes, are as the segments of the diameter, perpendicular thereto.*

For these areas are equal to the corresponding cylindrical surfaces, which are as the hights (Cor. 3. 4. VI).

Cor. 5. *The surface of any segment of the same sphere, is as the hight of the segment.*

Cor. 6. *The surface of the sphere is* $\frac{2}{3}$ *the whole surface of the circumscribing cylinder.*

For the two bases of the cylinder is half its curve surface (Cor. 3).

PROP. IV.

199 200 *The surface of the segment* BAD, *of a sphere; is equal to the area of a circle, whose radius is the cord* AB, *drawn from the vertex to the base.*

Let

Let C = circumference of the radius AB, and ABED = circumference of the ſphere. Then ſince the circumferences are as the radii (Cor. 9. IV), let $\frac{ABED}{AC} = \frac{C}{AB} = n$, or $ABED = n \times AC$, and $C = n \times AB$. Then the ſurface BAD $= AF \times ABED$ (Cor. 1. III) $= AF \times n \times AC = \frac{n \times AF \times AE}{2} = \frac{n \times AB^2}{2}$ (Cor. 17. IV) $= \frac{AB \times C}{2}$ = area of a circle whoſe radius is AB (34. IV). FIG. 199, 200

Cor. *The ſurface of the whole ſphere, is equal to the area of a circle, whoſe radius is the diameter* AE.

PROP. V.

The ſurface of a ſphere is double the curve ſurface of the inſcribed ſquare (or equilateral) cylinder EB. 201.

Draw the diameter ECB, then ED = DB. And ſince $EB^2 = ED^2 + DB^2$ (21. II) $= 2ED^2$; therefore circle EB = 2 circles ED (22. III). But ſurface of the ſphere = 4 circles EB (Cor. 3. III) = 8 circles ED. And $\frac{1}{2}$ED : AE or ED : : circle ED : curve ſurface AD (Cor. 2. 4. VI) = 4 circles ED. But 8 circles ED = twice 4 circles ED, or the ſurface of the ſphere = twice the curve ſurface of the cylinder.

Cor. 1. *The whole ſurface of the inſcribed cylinder is $\frac{3}{4}$ the ſurface of the ſphere.*

For the two baſes AB, ED = 2 circles ED, and the whole ſurface AD = 6 circles ED.

Cor. 2. *The curve ſurface of a cylinder, circumſcribing the ſphere, is double the curve ſurface of the inſcribed equilateral one. And the whole ſurface, is double to the whole ſurface.*

K For

FIG. 201. For the surface of the sphere = surface of the circumscribing cylinder (3). And the surface of the sphere = twice the surface of the inscribed one (5).

Again, the surface of the sphere = $\frac{2}{3}$ the whole surface of the circumscribing cylinder (Cor. 6. 3). And the surface of the sphere is = $\frac{4}{3}$ the whole surface of the inscribed cylinder (Cor. 1).

PROP. VI.

202. *The surface of any segment of a sphere* ABDC : *is to the curve surface of its inscribed cone* ABC :: *as the side of the cone* AB : *to the radius of the base* AO.

For if $n \times AB$ = circumference of the radius AB, and $n \times AO$ = circumference of the radius AO (Cor. 9. IV), then the circle AB = $\frac{AB^2 \times n}{2}$ (34. IV), and conic surface ABC = $\frac{AB \times n \times AO}{2}$ (Cor. 1. 5. VI). And the surface of the segment ABDC = circle AB (4); therefore surface of the segment ABDC : conic surface ABC :: $\frac{AB^2 \times n}{2}$: $\frac{AB \times n \times AO}{2}$ :: AB : AO (5. Proportion).

Cor. 1. *The surface of a hemisphere, is to the curve surface of its inscribed cone; as the diagonal of a square, to the side.*

For then AO, BO become radii of the sphere, and AB the diagonal.

Cor. 2. *If* ABC *be an equilateral cone, then the surface of the segment* ABDC *is twice the curve surface of the cone* ABC.

For then AB = AC = 2AO.

PROP. VII.

FIG. 203.

Let the cone DAE *be right-angled at* A. *Then the ſurface of the hemiſphere* BGE, *is to the curve ſurface of the right-angled circumſcribing cone* DAE; *as the ſide of a ſquare* AD, *is to the diagonal* DE.

Draw AC from the vertex of the cone A, to the center C; and CF ∥ to AE, or ⊥ to AD. Then AF = FD = FC = BC, and $CD^2 = CF^2 + FD^2 = 2BC^2$ (21. II). And the circle whoſe radius is CD = twice the circle whoſe radius is CB (Cor. 2. 35. IV) = ſurface of the hemiſphere BGE (Cor. 2. 3). Therefore the ſurface of the hemiſphere, or the circle whoſe radius is CD : ſurface of the cone DAE : : CD : AD (Cor. 3. 5. VI) : : AD : DE (20. II).

Cor. *The ſurface of a right-angled cone circumſcribing a hemiſphere, is double the ſurface of one inſcribed; taking either the curve ſurfaces, or the whole ſurfaces.*

For $\sqrt{2}$ × ſurface of the inſcribed cone = ſurface of the hemiſphere (Cor. 1. 6) = $\frac{1}{\sqrt{2}}$ × ſurface of the circumſcribing cone (7). Therefore the latter is = twice the former. And the baſe of the latter is likewiſe = twice the baſe of the former (by the demonſtration of this Prop.), therefore the whole is double to the whole.

PROP. VIII.

204.

The ſurface of the ſphere, is to the curve ſurface of an equilateral inſcribed cone BAD; *as* 8, *to* 3.

For ſince EF = $\frac{1}{4}$AF (Cor. 3. 41. IV), therefore ſurface BFD = $\frac{1}{4}$, and ſurface BAGD = $\frac{3}{4}$, the ſurface of the ſphere (Cor. 4. 3), = 2 curve ſurfaces

FIG. 204. surfaces of the cone BAD (Cor. 2. 6); or the surface of the cone $= \frac{3}{8}$ the surface of the sphere.

Cor. *The whole surface of an equilateral cone BAD, inscribed in a sphere, is $\frac{9}{16}$ of the sphere's surface.*

For $3BC^2 = BD^2$ (41. IV) $= 4BE^2$, and $BE^2 = \frac{3}{4}BC^2$, whence circle BD $= \frac{3}{4}$ circle BDG (35. IV) $= \frac{3}{16}$ the surface of the sphere (Cor. 3. 3); add this to the curve surface of the cone; then the whole surface of the cone $= \frac{3}{8} + \frac{3}{16}$ the sphere's surface $= \frac{9}{16}$ the surface of the sphere.

PROP. IX.

205. *The curve surface of an equilateral cone* ABD, *is to the surface of its inscribed sphere; as* 3 *to* 2.

Draw AE, CF $\perp$ to BD, BA; then by similar triangles AEB, AFC; $AE^2 : EB^2 :: AF^2 : FC^2$. But $AE^2 = \frac{3}{4}AB^2$ (39. II) $= 3AF^2$. Therefore $3AF^2$ $(AE^2) : AF^2 :: EB^2 : FC^2$ (4. Proportion) :: circle BD : circle FEG. But BE :: BA or 2BE :: circle BD : curve surface of the cone BAD (Cor. 3. 5. VI) $= 2$ circles BD; and circle FEG $= \frac{1}{4}$ surface of the sphere (Cor. 3. 3). Whence $3 : 1 :: 3AF^2 : AF^2 :: \frac{1}{2}$ surface of the cone : $\frac{1}{4}$ surface of the sphere. Therefore the surface of the sphere $= \frac{2}{3}$ the curve surface of the cone.

Cor. 1. *The surface of the sphere is $\frac{4}{9}$ the whole surface of the circumscribing equilateral cone.*

For the base BD $= \frac{1}{2}$ curve surface of the cone $= \frac{3}{4}$ surface of the sphere. Add this to the curve surface, which is $= \frac{3}{2}$ surface of the sphere; then the whole surface of the cone $= \frac{3}{2} + \frac{3}{4}$ the surface of the sphere $= \frac{9}{4}$ the surface of the sphere, or $\frac{4}{9}$ the whole surface of the cone $=$ the surface of the sphere.

Cor.

Cor. 2. *The curve ſurface of an equilateral cone inſcribed in a ſphere is* = $\frac{1}{4}$ *the curve ſurface of the circumſcribing equilateral one. And the whole ſurface of one* = $\frac{1}{4}$ *the whole ſurface of the other.* FIG. 205.

For $\frac{8}{3}$ the ſurface of the inſcribed cone = ſurface of the ſphere (8) = $\frac{2}{3}$ ſurface of the circumſcribed cone (9). Therefore the ſurface of the inſcribed = $\frac{1}{4}$ the ſurface of the circumſcribed one.

Alſo $\frac{4}{9}$ the whole ſurface of the circumſcribing one = ſurface of the ſphere (Cor. 1. 9) = $\frac{16}{9}$ the whole ſurface of the inſcribed cone (Cor. 8). Therefore the ſurface of the inſcribed cone = $\frac{1}{4}$ the ſurface of the circumſcribed cone.

Cor. 3. *The ſurfaces of a cylinder and equilateral cone, both circumſcribed about a ſphere, are as* 2 *to* 3; *both their curve ſurfaces and whole ſurfaces.*

For $\frac{2}{3}$ the curve ſurface of the cone = ſurface of the ſphere (9) = ſurface of the cylinder (3). Surface of the cylinder : ſurface of the cone : : 2 : 3.

Alſo $\frac{4}{9}$ the whole ſurface of the cone = ſurface of the ſphere (Cor. 1. 9) = $\frac{2}{3}$ the whole ſurface of the cylinder (Cor. 6. 3). Therefore, whole ſurface of the cylinder : whole ſurface of the cone : : $\frac{4}{9}$: $\frac{2}{3}$ or $\frac{6}{9}$: : 2 : 3.

SCHOLIUM.

From the foregoing propoſitions are deduced, the proportion of the ſphere's ſurface, to the ſurfaces of the inſcribed and circumſcribed equilateral cylinder and cone, as follows:

FIG. 205.

Surface of the ſphere	—— ——	16
Inſcribed cylinder's curve ſurface	—	8
—— —— whole ſurface	—	12
Circumſcribed cylinder's curve ſurface		16
—— —— whole ſurface	—	24
Inſcribed cone's curve ſurface	——	6
—— —— whole ſurface	—	9
Circumſcribing cone's curve ſurface	—	24
—— —— whole ſurface	—	36.

PROP. X.

206. *A ſphere is equal to a cone whoſe hight is the radius* AC, *and baſe the ſurface of the ſphere* AEF.

Take three points in the ſurface of the ſphere, as A, B, D, extremely near together, forming the ſmall triangle ABD, on the ſurface of the ſphere. Let a plane paſs through theſe three points A, B, D; the ſmall portion of which ABD will coincide with a portion of the ſpherical ſurface ABD, extremely near. And the radius CA will be ⊥ thereto (2). Therefore the portion of the ſphere CABD is nothing but the pyramid whoſe baſe is ABD, a ſmall part of the ſphere's ſurface, and hight the radius CA. In like manner the whole ſphere may be divided into ſmall pyramids, ſuch as CABD, whoſe baſe is a ſmall portion of the ſpherical ſurface; and common altitude, the radius CA. Therefore the ſum of all theſe pyramids CABD, make up the ſphere; and the ſum of all the baſes ABD, make up the ſpherical ſurface. That is, the ſphere is equal to the ſum of all theſe pyramids, whoſe baſes are all the parts of the ſurface, of the ſphere, and common altitude the radius CA; and that is equal to one pyramid or cone, whoſe baſe is the ſurface of the ſphere, and hight the radius (Ax. 2).

Cor.

Cor. 1. *A sphere is equal to a cone, whose hight is the radius, and base equal to four great circles of the sphere.* FIG. 206.

For the surface of the sphere is equal to four great circles (Cor. 3. 3).

Cor. 2. *A sphere is equal to a cone whose hight is twice the diameter, and base, a great circle of the sphere.*

By Cor. 4. 20. VI.

Cor. 3. A *hemisphere is double its inscribed cone.*

For a hemisphere = a cone whose base is a great circle, and hight equal to the diameter (Cor. 2); and that is double to a cone of the same base, and half the hight (Cor. 1. 20. VI).

PROP. XI.

Any sphere BANR, *is* $\frac{2}{3}$ *its circumscribing cylinder,* DM. 207.

Let AC be the axis of the hemisphere BAN. From the center C, draw the diagonal CD; and draw PL ⊥ to AC, and OH parallel to it, and exceeding near it. Then if the figure ADBC revolve round the axis AC; then ADBC will generate the cylinder BDGN; the quadrant BVA, the hemisphere BAN; and ADC, the cone ADCG. Then $VC^2 = VL^2 + LC^2$ (21. II); that is, $PL^2 = VL^2 + KL^2$ (for DA = AC, and KL = LC (13. II). Therefore the circle described by LP = the two circles described by LV and LK (Cor. 2. 35. IV). Take away the circle described by LV, from both, and there remains the annulus or ring described by VP = circle described by LK. For the same reason the annulus described by OI = circle described by FH. Therefore the small prismatic solid contained between PN and OI, quite round the figure = cone frustum contained between KL and FH, round the figure

FIG. 207. figure (12. VI). In like manner every part of the figure BDAVB = correspondent part of DACG. Therefore the total sum of the first = total sum of the last, that is, the solid BDAGNAVB = cone DCG (Ax. 2) = $\frac{1}{3}$ the cylinder DBNG (20. VI). Therefore the remaining part, or the hemisphere BAN = the remaining $\frac{2}{3}$ of the cylinder BDGN. Whence the double thereof, or the whole sphere ABRN = $\frac{2}{3}$ of the whole cylinder EG.

Otherwise.

The cone whose base is BN, and hight CA, or the cone DCG = half the hemisphere BAN (Cor. 3. 10). And the same cone DCG = $\frac{1}{3}$ the cylinder BDGN, (20. VI). Therefore $\frac{1}{2}$ hemisphere = $\frac{1}{3}$ cylinder, and the hemisphere = $\frac{2}{3}$ cylinder BG. Whence the whole sphere = $\frac{2}{3}$ the cylinder EG.

Cor. 1. *The concave solid* BFADBER *&c.* = $\frac{1}{2}$ *the sphere* BANR.

208. Cor. 2. *A right cone, sphere, and cylinder, all of the same diameter and hight, are as* 1, 2, 3 *respectively; or* ABD : AHGI : EBDF : : 1 : 2 : 3.

PROP. XII.

206. *The sector of a sphere* CGAH, *is equal to a cone whose hight is the radius; and base, the surface of the sector* GAH.

This is demonstrated as Prop. X. For if the sector be divided into a multitude of extremely small sectors CABD, the base of each will be a small portion of the spherical surface ABD. And as all the pyramids make up the sector, and are the elements thereof; so all the bases are the elements of the surface GAH, and make it up. And as the hights of all the pyramids is the same, they are all equal to one pyra-

pyramid of the same hight, and base the sum of all the bases (Cor. 1. 18. VI). That is, the sector CGAH = a pyramid or cone whose hight is the radius, and base the surface GAH. FIG. 206.

Cor. 1. *The sector of a sphere,* CGAH = *a cone, whose hight is the radius* AC; *and base a circle whose radius is* AG. *And the sector* CGBH = *a cone whose radius is* CB, *and base a circle whose radius is* BG.

For the surface GAH = a circle whose radius is AG (4); and the surface GBH = a circle whose radius is BG (ibid.).

Cor. 2. *Sectors of spheres, are to one another, in the complicate ratio of their surfaces and radii.*

For the cones, equal thereto, are as the bases and hights (Cor. 3. 20. VI).

PROP. XIII.

If it be made, as BD : BA : : *radius* CA : CF; *then the cone* GFH *is equal to the segment of the sphere,* GAH. 210.

Draw CG, BG and FCB; then CA : CF : : BD : BA (hyp.) : : $BD^2 : BG^2$ (Cor. 1. 20. II) : : $GD^2 : GA^2$ (20. II) : : circle GD (or circle whose radius is GD) : circle GA (35. IV). Therefore the cone whose hight is CF, and base the circle GD = cone whose hight is CA, and base the circle GA (Cor. 4. 20. VI) = sector CGAH (Cor. 1. XII). Subtract, or add the cone GCH, on the same base GH, and then the cone GFH = segment GAH.

Cor. 1. *If* BD : DA : : *radius* CA : AF. *Then the cone* GFH = *segment* GAH.

For since BD : BA : : CA : CF, therefore BD : BA — BD : : CA : CF — CA (13. Proportion); that is, BD : DA : : CA : AF.

Cor. 2. *The segment* GAH, *is to the inscribed cone* GAH; *as* FD *to* AD. Cor.

FIG. 210. Cor. 3. *The segment* GAH : *segment* GBH :: $\overline{GC + DB} \times AD^2 : \overline{GC + AD} \times DB^2$.

For the hight of the cone, equal to the segment GAH, that is, $DF = \frac{GC}{DB} \times DA + DA$ (Cor. 1) $= \frac{GC + DB}{DB} \times DA$. And in like manner, the hight of the cone equal to the segment GBH, is $\frac{GC + DA}{DA} \times DB$. And these cones are as the altitudes (Cor. 1. 20. VI); that is, as $\frac{GC + DB}{DB} \times DA$, and $\frac{GC + DA}{DA} \times DB$, or as $\overline{GC + DB} \times DA^2 : \overline{GC + DA} \times DB^2$.

PROP. XIV.

210. *The segment of a sphere* GAH, *is equal to a cone, whose hight is* AD, *the hight of the segment; and base,* $\frac{3}{2}$ *the base of the segment* GH, *together with* $\frac{1}{2}$ *a circle whose radius is the hight of the segment* AD.

Let $\odot AG$ denote the circle whose radius is AG, and so of the rest. Then segment GAH = sector CGAH $\mp$ cone GCH (fig. 1, 2) $= \frac{1}{3}AC \times \odot AG \mp \frac{1}{3}CD \times \odot GD$ (Cor. 1. 12); and 3 segments GAH $= AC \times \odot AG \pm \overline{AD \mp AC} \times \odot GD = AC \times \overline{\odot AG - \odot GD} + AD \times \odot GD = AC \times \odot AD + AD \times \odot GD$ (Cor. 2. 35. IV).

But $AD : AB :: AD^2 : AG^2$ (Cor. 1. 20. II) $:: AD^2 : AD^2 + DG^2 :: \odot AD : \overline{\odot AD + \odot DG}$ (Cor. 2. 35. IV), therefore $AD \times \overline{\odot AD + \odot DG} = AB \times \odot AD = 2AC \times \odot AD$, and $AD \times \overline{3\odot DG + \odot AD} = 2AC \times \odot AD + 2AD \times \odot GD$. And $AD \times \overline{1\frac{1}{2}\odot GD + \frac{1}{2}\odot AD} = AC \times \odot AD + AD \times \odot GD = 3$ segments GAH.

Corol-

Corollary. *The ſegment* GAH $= \frac{1}{6}AD \times \overline{3\Theta GD + \Theta AD}$. FIG. 210.

PROP. XV.

The fruſtum or middle zone of a ſphere ZGHF, *is equal to a cone whoſe hight is the hight of the zone* CD; *and baſe, two great circles* ZF, *together with the leſſer baſe* GH. 211.

For the zone ZH = hemiſphere ZAF — the ſector CGAH + the cone GCH $= AC \times \frac{2}{3}\Theta ZC$ (11) — $AC \times \frac{1}{3}\Theta AG$ (Cor. 1. 12) + $CD \times \frac{1}{3}\Theta GD$ (20. VI) $= AD \times \frac{2}{3}\Theta ZC + DC \times \frac{2}{3}\Theta ZC - AC \times \frac{1}{3}\Theta AG + CD \times \frac{1}{3}\Theta GD$. But $AD : AC :: AG^2 : AZ^2$ (18. IV) $:: AG^2 : 2AC^2$ (21. II) $:: \Theta AG : 2\Theta ZC$ (35. IV). Therefore $AD \times 2\Theta ZC = AC \times \Theta AG$. And $AD \times \frac{2}{3}\Theta ZC = AC \times \frac{1}{3}\Theta AG$. Therefore the zone $ZH = DC \times \frac{2}{3}\Theta ZC + DC \times \frac{1}{3}\Theta GD = \frac{1}{3}DC \times \overline{2\Theta ZC + \Theta GD}$.

Cor. *The zone* ZH *is equal to* $\frac{1}{3}DC \times$ *twice the circle* ZF + *the circle* GH.

PROP. XVI.

An orb or hollow ſphere is equal to the fruſtum of a cone, whoſe greater baſe is the ſurface of the greater ſphere; and leſſer baſe, the ſurface of the leſſer: and hight, the difference of the radii.

For the orb is equal to the difference of the two ſpheres; that is, to the difference of two cones whoſe hights are the radii of the ſpheres, and baſes the ſurfaces (10).

PROP.

FIG.

PROP. XVII.

212. *The surfaces of spheres* GH, IK, *are as the squares*
213. *of the diameters,* AB, DF.

For the surface of the sphere GH = 4 circles AGBH, and the surface of the sphere IK = 4 circles IDKF (Cor. 3. III). But 4 circles AGBH : 4 circles DIFK : : circle AGBH : circle DIFK (Cor. 1. 5. Proportion) : : $AB^2 : DF^2$ (35. IV).

PROP. XVIII.

Spheres GH, IK, *are to one another, as the cubes of their diameters,* AB, DF.

For the sphere GH = $\frac{2}{3}$ the cylinder, whose base is AGBH, and hight AB. And the sphere IK = $\frac{2}{3}$ the cylinder, whose base is DIFK, and hight DF (11). Therefore sphere GH : sphere IK : : $\frac{2}{3}$AGBH × AB : $\frac{2}{3}$DIFK × DF (Cor. 3. 19. VI) : : AGBH × AB : DIFK × DF (5. Proportion) : : $AB^2 \times AB : DF^2 \times DF$ (35. IV. and 7. Proportion) : : $AB^3 : DF^3$.

PROP. XIX.

214. *Similar solids inscribed in spheres* GH, IK, *are as the*
215. *cubes of the diameters of the spheres* AB : DF.

From any two equal and correspondent angles A, D, draw the diameters AB, DF. Then since the solids are inscribed after a similar manner in respect of the diameters AB, DF. It will be AG : DI : : AB : DF (19. III). But solid AE : solid DL : : $AG^3 : DI^3$ (24. VI) : : $AB^3 : DF^3$ (Cor. 3. 18. Proportion).

Cor. 1. *Similar solids inscribed in spheres, are as the spheres.*

For spheres are also as the cubes of their diameters (18).

Cor.

Cor. 2. *The surfaces of similar solids inscribed in spheres, are as the squares of the diameters of the spheres.* FIG. 214. 215.

For surface of AE : surface of DL : : $AG^2 : DI^2$ (7. VI) : : $AB^2 : DF^2$.

Cor. 3. *The surfaces of similar solids inscribed in spheres, are as the surfaces of the spheres.*

For they are both as the squares of the diameters (17).

PROP. XX.

A sphere, is to any circumscribing solid BF, *(all whose planes touch the sphere)*; *as the surface of the sphere, to the surface of the solid.* 216.

Since all the planes touch the sphere, the radius drawn to all the points of contact, will be $\perp$ to each plane (2). Therefore if planes be drawn through the center C of the sphere, and through all the sides of the body; then the body will be divided into pyramids, BCAE, BCAD, *&c.* whose bases are the planes BAE, BAD, *&c.*; and their common altitude CP, the radius of the sphere. And the sum of all these pyramids, or the whole solid, is equal to a pyramid or cone, whose base is the sum of all the plane figures, and hight the radius CP (Cor. 1. 18. and Cor. 2. 20. VI). But the sphere is also equal to a cone or pyramid whose base is the surface of the sphere, and hight the same radius CP (10). And this last cone : former cone : : base of the latter : base of the former, (Cor. 2. 20. VI.); that is, the sphere : circumscribing solid : : surface of the sphere : surface of the solid.

Cor. 1. *All circumscribing cylinders, cones,* &c. *are to the sphere, as their surfaces are.*

For

FIG. 216. For any cylinder, or cone, may be conceived to be made up of an infinite number of ſmall planes, all of which touch the ſphere.

Cor. 2. *All bodies circumſcribing the ſame ſphere, are to one another as their ſurfaces.*

Cor. 3. *The ſphere is the greateſt or moſt capacious of all bodies of equal ſurface.*

For if the planes be ſuppoſed to touch the ſphere, their areas will be greater than the ſurface of the ſphere, which is contrary to the hypotheſis; therefore the planes muſt fall within the ſphere; and then the perpendicular upon them will be ſhorter than the radius, and therefore the body will be leſs than the ſphere, as having the ſame baſe, and a leſs hight.

PROP. XXI.

210. *Any ſegment of a ſphere* GAH, *is to its inſcribed cone; as* BC + BD, *to* BD.

For if AF $= \frac{AD}{DB} \times AC$, then the ſegment GAH = cone GFH (Cor. 1. 13). Therefore FD $= \frac{AD}{DB} \times AC + AD$. And this cone GFH : cone GAH :: DF : DA (Cor. 1. 20. VI) :: $\frac{AC}{BD} \times AD + AD : AD :: \frac{AC + BD}{BD} \times AD : AD ::$ AC + BD : BD (5. Proportion).

Cor. 1. *A hemiſphere is double the inſcribed cone.*

For then BD = AC or BC.

Cor. 2. *The ſegment containing an equilateral cone, is equal to 3 times the cone.*

For then BD = $\frac{1}{2}$BC (Cor. 3. 41. IV).

PROP.

PROP. XXII.

FIG. 203.

If the cone DAE *circumscribing a hemisphere be right-angled at* A; *that cone* DAE *is to the inscribed hemisphere; as* $\sqrt{2}$ *to* 1.

For let $\odot$ stand for circle, then supposing the same construction as in Prop. VII, then we have $CD^2 = 2BC^2$, and $CD = BC\sqrt{2} = DF\sqrt{2}$, and $CD : DF :: \sqrt{2} : 1$ (Cor. 1. 12. Proportion); also $\odot CD = 2\odot CB$, and $AC = CD$. The cone DAE $= \odot CD \times \frac{1}{3}AC$ (20. VI) $= 2\odot CB \times \frac{1}{3}CD$. Also the hemisphere $= \frac{2}{3}\odot CB \times GC$. (11) $= 2\odot CB \times \frac{BC}{3}$. Therefore the cone : hemisphere $:: 2\odot CB \times \frac{1}{3}CD : 2\odot CB \times \frac{1}{3}BC :: CD : CB$ or $DF :: \sqrt{2} : 1$.

Cor. *A right-angled cone, circumscribing a hemisphere, is to the inscribed cone; as* $2\sqrt{2}$ *to* 1.

For the circumscribed cone : hemisphere $:: \sqrt{2} : 1 :: 2\sqrt{2} : 2$ (22).

And hemisphere : inscribed cone $:: 2 : 1$ (Cor. 1. 21).

Therefore circumf. cone : inf. cone $:: 2\sqrt{2} : 1$ (15. Proportion).

PROP. XXIII.

201.

A sphere is to its inscribed equilateral cylinder AD, *as* $4\sqrt{2}$ *to* 3.

Draw the diameter BE, then $BE^2 = DE^2 + DB^2$ (21. II) $= 2DE^2$, and circle AEDB $= 2$ circles BD (35. IV); also $BE = DE\sqrt{2} = BD\sqrt{2}$. Now The sphere $= \frac{2}{3}AEDB \times BE$ (11) $= \frac{2}{3}AEDB \times BD\sqrt{2}$, the cylind. $=$ circle $ED \times BD = \frac{1}{2}AEDB \times BD$. Then sphere : cylinder $:: \frac{2}{3}AEDB \times BD\sqrt{2} : \frac{1}{2}AEDB \times BD :: \frac{2}{3}\sqrt{2} : \frac{1}{2} :: 4\sqrt{2} : 3$.

Cor.

FIG. 201. Cor. *The circumscribed equilateral cylinder, is to the inscribed equilateral cylinder; as* $2\sqrt{2}$ *to* 1.

For $\frac{2}{3}$ the circumscr. cylinder = sphere (11) = $\frac{4\sqrt{2}}{3}$ × the inscr. cylinder. Therefore the circumsc. cylinder = $2\sqrt{2}$ × inscr. cylinder.

PROP. XXIV.

204. *The sphere is to the inscribed equilateral cone* BAD, *as* 32 *to* 9.

Let ⊙BE denote the circle whose radius is BE, &c. then BD^2 or $4BE^2 = 3BC^2$ (41. IV), and $BE^2 = \frac{3}{4}BC^2$, and ⊙BE = $\frac{3}{4}$⊙BC (Cor. 1. 35. IV). Also AE = $\frac{3}{2}$AC (Cor. 3. 41. IV). Then the sphere = $\frac{2}{3}$⊙BC × 2AC (11). And cone = ⊙BE × $\frac{1}{3}$AE (20. VI) = $\frac{3}{4}$⊙BC × $\frac{1}{3}$ × $\frac{3}{2}$AC. Therefore, sphere : cone : : $\frac{2}{3}$⊙BC × 2AC : $\frac{3}{4}$⊙BC × $\frac{1}{2}$AC : : $\frac{4}{3}$: $\frac{3}{8}$: : 32 : 9.

PROP. XXV.

205. *A sphere is to its circumscribed equilateral cone* ABD, *as* 4 *to* 9.

The construction of Prop. IX. remaining; let ⊙FC denote the circle whose radius is FC, &c. Then $EB^2 = 3FC^2$, and ⊙BE = 3⊙FC (35. IV), and CF or CE = $\frac{1}{2}$CA (Cor. 31. II), and AE = 3CF.

The sphere = $\frac{2}{3}$⊙CF × 2CF (11).

The cone = ⊙BE × $\frac{1}{3}$ AE (20. VI) = 3⊙FC × FC.

Therefore sphere : cone : : $\frac{2}{3}$⊙CF × 2CF : 3⊙CF × CF : : $\frac{4}{3}$: 3 : : 4 : 9.

Cor. 1. *The circumscribed equilateral cone is eight times the inscribed equilateral cone.*

For the circumscr. cone : sphere : : 9 : 4.
And sphere : inscr. cone : : 32 : 9 (24).
Therefore circumscr. cone : inscr. cone : : 32 : 4 (15. Proportion) : : 8 : 1.

Cor.

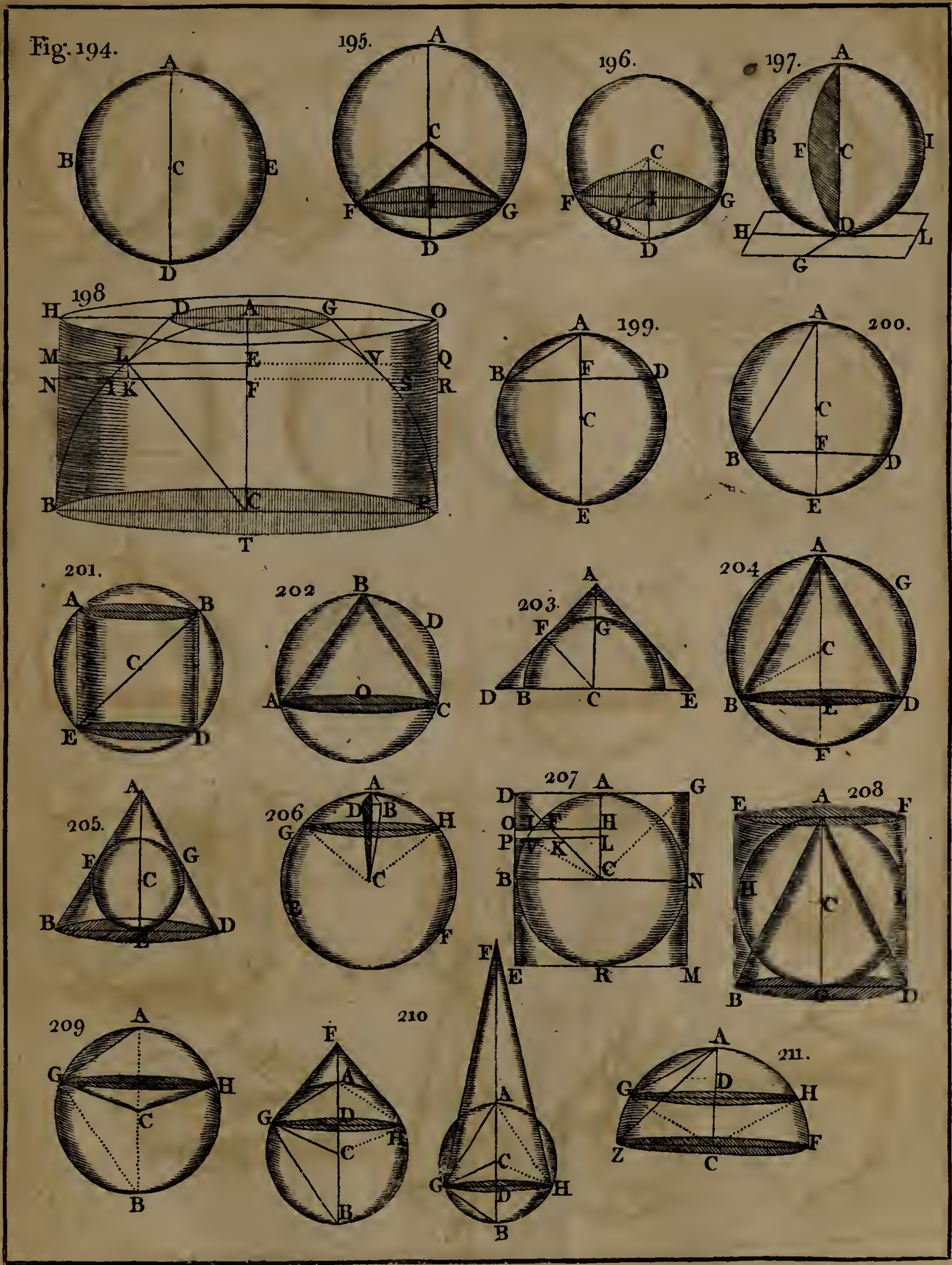


Fig. 194.
195.
196.
197.
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211.

Cor. 2. *The circumscribed cylinder is $\frac{2}{3}$ the circumscribed equilateral cone.* FIG. 205.

For $\frac{2}{3}$ the cylinder = ſphere (11) = $\frac{4}{9}$ the cone; and the cylinder = $\frac{2}{3}$ the cone.

Cor. 3. *The ſphere* EF *is to the circumſcribing right cylinder* BC, *and this cylinder to the circumſcribing equilateral cone* ADG, *as* 2 *to* 3; *both in reſpect of their whole ſurfaces and ſolidities.* 217.

This appears from Cor. 6. 3. and Cor. 3. 9. and Prop. 11. and Cor. 2. 25.

Cor. 4. *The circumſcribing right cylinder, and equilateral cone, are to one another as* 2 *to* 3; *both in regard to their curve ſurfaces, their whole ſurfaces, ſolidities, baſes, and hights.*

As to the ſurfaces it appears by Cor. 3. 9; and the ſolidities, by Cor. 3. of this. As to the baſes, ſince ⊙BE = 3⊙FC (fig. 205), or ⊙FC = $\frac{1}{3}$⊙BE, therefore 2⊙FC = $\frac{2}{3}$⊙BE, or the two baſes of the cylinder = $\frac{2}{3}$ the baſe of the cone.

And for the hight, AE = 3CF, or 2CF = $\frac{2}{3}$AE; that is, the hight of the cylinder = $\frac{2}{3}$ the hight of the cone.

SCHOLIUM.

From the foregoing propoſitions, is eaſily deduced the proportion which the ſphere has to the inſcribed and circumſcribed equilateral cylinders and cones, as follows:

Solidity of the ſphere	32
———— inſcribed cone	9
———— inſcribed cylinder	$12\sqrt{2}$
———— circumſcribed cylinder	48
———— circumſcribed cone	72.

L PROP.

PROP. XXVI.

FIG. 218. *The square of the side of a regular pyramid inscribed in a sphere, is* $\frac{2}{3}$ *the square of the diameter:* $AE^2 = \frac{2}{3}EF^2$.

For drawing ECF $\perp$ to the base ABD, $3AC^2 = AB^2$ (41. IV) $= AE^2 = AC^2 + CE^2$ (21. II); and $2AC^2 = CE^2$, or $\frac{1}{2}CE^2 = AC^2 = EC \times CF$ (17. IV), therefore $CF = \frac{1}{2}CE$, and $EF = \frac{3}{2}CE$, or $CE = \frac{2}{3}EF$, and $AC^2 = \frac{1}{2}CE^2 = \frac{2}{9}EF^2$. Therefore $AE^2 = AC^2 + CE^2$ (21. II) $= \frac{2}{9}EF^2 + \frac{4}{9}EF^2 = \frac{6}{9}EF^2 = \frac{2}{3}EF^2$.

Cor. 1. *The hight of the pyramid is* $\frac{2}{3}$ *the diameter of the sphere,* $EC = \frac{2}{3}EF$.

Cor. 2. *The diameter of the sphere : diameter of the circle comprehending the base of pyramid : : as* $3 : \sqrt{8}$.

For $AC^2 = \frac{2}{9}EF^2$, and $4AC^2 = \frac{8}{9}EF^2$.

Cor. 3. *The area of the base* $ADB = EF^2 \times \frac{\sqrt{3}}{6}$.

For the area $ADB = \frac{AB^2}{4}\sqrt{3}$ (39. II). And AB^2 or $AE^2 = \frac{2}{3}EF^2$. Therefore $ADB = \frac{1}{6}EF^2\sqrt{3}$.

Cor. 4. *The radius of the inscribed sphere* $= \frac{1}{6}EF$.

For it is $= EC - \frac{1}{2}EF = \frac{1}{6}EF$.

PROP. XXVII.

218. *The solidity of a regular pyramid inscribed in a sphere, is* $\frac{1}{27}EF^3\sqrt{3}$.

For the solidity $= \frac{1}{3}EC \times$ base ABD (18. VI) $= \frac{2}{9}EF \times ABD$ (Cor. 1. 26) $= \frac{2}{9}EF \times \frac{1}{6}EF^2\sqrt{3}$ (Cor. 3. 26) $= \frac{\sqrt{3}}{27} \times EF^3$.

PROP.

PROP. XXVIII. FIG. 219.

The square of the diameter of a sphere, is thrice the square of the side of its inscribed cube: $FA^2 = 3FD^2$.

Through the opposite sides AG, DF, suppose the plane FDAG to be drawn; and through two opposite angles A, F, draw the diameter of the sphere AF. Then $DA^2 = DB^2 + BA^2 = 2DB^2 = 2DF^2$ (21. II). Also $FA^2 = FD^2 + DA^2 = FD^2 + 2FD^2 = 3FD^2$ (ibid.).

Cor. 1. *The side of the cube* $DF = \frac{1}{3}FA\sqrt{3}$.

Cor. 2. *The diameter of the sphere* AF, *is to the diameter* DA *of the circle comprehending one face of the cube; as* 1 *to* $\frac{1}{3}\sqrt{6}$.

For $FA = FD \times \sqrt{3}$, and $DA = FD\sqrt{2}$; and $FD\sqrt{3} : FD\sqrt{2} :: 1 : \sqrt{\frac{2}{3}}$, or $1 : \frac{1}{3}\sqrt{6}$.

Cor. 3. *The area of one face of the cube* DBAI *is equal to* $\frac{1}{3}FA^2$.

Cor. 4. *The sum of the squares of the sides of the inscribed pyramid and cube, is equal to the square of the diameter.*

For the former is $\frac{2}{3}$, and the latter $\frac{1}{3}$, of the square of the diameter (26 and 28).

Cor. 5. *The diameter of the circle containing one face of the cube* DA, *is equal to the side of the pyramid.*

For $DA^2 = 2DF^2 = \frac{2}{3}FA^2$ (28) = square of the side of the pyramid (26).

Cor. 6. *The radius of the inscribed sphere is* $\frac{1}{2}$ *the side* FD.

FIG.

PROP. XXIX.

219. *The solidity of a cube inscribed in a sphere, is* $\frac{\sqrt{3}}{9}$ *multiplied into the cube of the diameter of the sphere:* $\frac{\sqrt{3}}{9} \times AF^3$.

For $DF = FA\sqrt{\frac{1}{3}}$, and $DF^3 = FA^3 \times \frac{1}{3}\sqrt{\frac{1}{3}} = FA^3 \times \frac{1}{9}\sqrt{3}$ (28).

Cor. *The inscribed cube is thrice the inscribed pyramid.*

PROP. XXX.

220. *The square of the diameter of a sphere is double to the square of the side of an inscribed regular octaedron* ABFDEG: $AG^2 = 2AB^2$.

Through two opposite angles A, G, draw the diameter AG; then the angle ABG is right (14. 4); therefore $AG^2 = AB^2 + BG^2 = 2AB^2$ (21. II).

Cor. 1. *The square of the diameter of a circle comprehending a triangle of the octaedron, is* $\frac{2}{3}$ *the square of the diameter of the sphere.*

For AB^2 = thrice the square of the radius (41. IV) = $\frac{3}{4}$ the square of the diameter, and $AB^2 = \frac{1}{2}AG^2$ (30); therefore the diameter square $= \frac{2}{3}AG^2$.

Cor. 2. *The diameter of a circle containing the triangle of the octaedron, is equal to the side of the pyramid.*

Cor. 3. *The same circle comprehends both the square of the cube, and the triangle of an octaedron, inscribed in the same sphere.*

For

For the former diameter is $\frac{1}{3}\sqrt{6}$, and the latter $\sqrt{\frac{2}{3}} = \frac{1}{3}\sqrt{6}$. FIG. 220.

Cor. 4. *The area of one of the faces of the octaedron, as* ABE, *is* $\frac{\sqrt{3}}{8}$ *multiplied into the square of the diameter of the sphere*; $= \frac{\sqrt{3}}{8} \times AG^2$.

For the triangle ABE $= \frac{AB^2}{4}\sqrt{3}$ (39. II) $= \frac{AG}{8}\sqrt{3}$.

Cor. 5. *The radius of the inscribed circle is* $\frac{1}{3}$ AB $\sqrt{6}$.

For it is the perpen. from C upon ABE, suppose it $= P$, $D =$ diameter of the circle encompassing ABE. Then $PP = \frac{1}{4}AG^2 - \frac{1}{4}DD$; and $4PP = AG^2 - D^2 = AG^2 - \frac{2}{3}AG^2$ (Cor. 1, 30) $= \frac{1}{3}AG^2 = \frac{2}{3}AB^2$, and $2P = AB\sqrt{\frac{2}{3}} = \frac{1}{3}AB\sqrt{6}$. and $P = \frac{1}{6}AB\sqrt{6}$.

PROP. XXXI.

The solidity of an octaedron BD, *inscribed in a sphere, is* $\frac{1}{6}$ *the cube of the diameter of the sphere* AG.

For the body consists of two pyramids BEDFA and BEDFG, standing on the square base BEDF, 220.
Therefore the solidity $= DE^2 \times \overline{\frac{1}{3}AC + \frac{1}{3}CG} = \frac{1}{2}BD^2 \times \frac{AG}{3} = \frac{1}{6}AG^3$.

Cor. *A sphere, is to the inscribed octaedron; as the circumference of the sphere, to its diameter.*

For the sphere is $= \frac{2}{3}$ the circle ABGD $\times$ AG (11) $= \frac{2}{3}AG \times$ circumference ABGD $\times \frac{1}{4}AG$ (34. IV). Therefore sphere : octaedron :: ABGD $\times \frac{1}{6}AG^2 : \frac{1}{6}AG^3$ :: ABGD : AG.

FIG. 221.

PROP. XXXII.

The square of the diameter of a sphere, is to the square of the side of its inscribed regular dodecaedron DA; *as* 6 *to* $3 - \sqrt{5}$; *or as* $9 + 3\sqrt{5}$, *to* 2.

Let A be a solid angle of the dodecaedron; AG, AI, AL, three pentagons forming the $\angle$A. Draw the diagonals, BD, BF, DF. And on the plane BDF let fall the perp. AC, and draw DC. Then $DF^2 = 3DC^2$ (41. IV), and $DC^2 = \frac{1}{3}DF^2$, and $CA^2 = DA^2 - DC^2$ (Cor. 1. 21. II) $= DA^2 - \frac{1}{3}DF^2 = DA^2 - \frac{1}{3}DA^2 \times \frac{3 + \sqrt{5}}{2}$ (Cor. 3. 43. IV) $= \frac{3 - \sqrt{5}}{6}DA^2$, therefore $CA = \frac{\sqrt{3 - \sqrt{5}}}{\sqrt{6}}DA$. But $\frac{DA^2}{CA}$ = diameter of the sphere (Cor. 17. IV), or the diameter $= \frac{DA^2 \times \sqrt{6}}{DA \times \sqrt{3 - \sqrt{5}}} = \frac{\sqrt{6}}{\sqrt{3 - \sqrt{5}}} \times DA = S$; and diameter square, $SS = \frac{6DA^2}{3 - \sqrt{5}}$; and $DA^2 = \frac{3 - \sqrt{5}}{6}SS = \frac{2SS}{9 + 3\sqrt{5}}$.

Cor. 1. *The square of the diameter of the sphere, is to the square of the diameter of the circle containing one face of the dodecaedron* AL; *as* 15 *to* $10 - 2\sqrt{5}$.

Let S = diameter of the sphere, R = radius of the circle circumscribing the pentagon, then $AD^2 = \frac{3 - \sqrt{5}}{6}SS$ (32); and $RR = \frac{2AD^2}{5 - \sqrt{5}}$ (44. IV) $= \frac{2}{5 - \sqrt{5}} \times \frac{3 - \sqrt{5}}{6} SS = \frac{1}{3}SS \times \frac{3 - \sqrt{5}}{5 - \sqrt{5}} = \frac{1}{3}SS \times \frac{3 - \sqrt{5}}{5 - \sqrt{5}} \times \frac{5 + \sqrt{5}}{5 + \sqrt{5}} = \frac{1}{3}SS \times$

15 +

$\frac{15 + 3\sqrt{5} - 5\sqrt{5} - 5}{25 - 5} = \frac{1}{3}SS \times \frac{10 - 2\sqrt{5}}{20} =$ FIG. 221.

$\frac{5 - \sqrt{5}}{30}SS$, and the square of the diameter of that circle or $4RR = \frac{10 - 2\sqrt{5}}{15}SS$.

Cor. 2. *The area of one pentagon of the dodecaedron, is equal to* $\frac{5}{12}\sqrt{\frac{5 - \sqrt{5}}{10}}$ *multiplied by the square of the diameter of the sphere.*

For let O be the center of the circle circumscribing the pentagon AI; and OP $\perp$ to FI. Then $OP^2 = \frac{3 + \sqrt{5}}{8} \times RR$ (Cor. 1. 44. IV) $= \frac{3 + \sqrt{5}}{8} \times \frac{5 - \sqrt{5}}{30}SS$; and the area FOI $= \frac{1}{2}OP \times FI = \frac{SS}{2}\sqrt{\frac{3 + \sqrt{5}}{8} \times \frac{5 - \sqrt{5}}{30} \times \frac{3 - \sqrt{5}}{6}} = \frac{SS}{2}\sqrt{\frac{4}{48} \times \frac{5 - \sqrt{5}}{30}}$; and since there are 5 such triangles in the pentagon, the pentagon $= \frac{5}{2}SS\sqrt{\frac{1}{12} \times \frac{5 - \sqrt{5}}{30}} = \frac{5SS}{12}\sqrt{\frac{5 - \sqrt{5}}{10}}$.

Cor. 3. *The side of the cube is equal to the diagonal* DF, *of the pentagon of a dodecaedron inscribed in the same sphere.*

For $DA^2 = \frac{3 - \sqrt{5}}{6}SS$ (32), and $DF = \frac{1 + \sqrt{5}}{2}DA$ (Cor. 3. 43. IV); and $DF^2 = \frac{6 + 2\sqrt{5}}{4}DA^2 = \frac{3 + \sqrt{5}}{2}DA^2 = \frac{3 + \sqrt{5}}{2} \times \frac{3 - \sqrt{5}}{6}SS = \frac{9 - 5}{2 \times 6}SS = \frac{4}{2 \times 6}SS = \frac{1}{3}SS$. But the square of the side of the inscribed cube is also $= \frac{1}{3}SS$ (28). Therefore

FIG. 221. Therefore the diagonal in the pentagon = side of the cube.

PROP. XXXIII.

The cube of the diameter of a sphere, is to the solidity of the inscribed regular dodecaedron; as 1, *to* $\frac{5}{6}\sqrt{\frac{3+\sqrt{5}}{30}}$.

Let S = diameter of the sphere, R = radius of the circle encompassing the pentagon, P = perpendicular from the center of the sphere upon the pentagon, then $RR = \frac{5-\sqrt{5}}{30}SS$ (Cor. 1. 32). Then $PP = \frac{1}{4}SS - RR$ (Cor. 1. 21. II) $= \frac{15-10+2\sqrt{5}}{60}SS = \frac{5+\sqrt{5}}{60}SS$, and $P = S\sqrt{\frac{5+2\sqrt{5}}{60}}$, and the area of the pentagon $= \frac{5}{12}SS\sqrt{\frac{5-\sqrt{5}}{10}}$ (Cor. 2. 32). Therefore the pyramid whose base is the pentagon, and vertex at the center of the sphere, is $= \frac{1}{3}S^3 \times \frac{5}{12}\sqrt{\frac{5+2\sqrt{5}}{60}} \times \frac{5-\sqrt{5}}{10}$ (18. VI) $= \frac{5}{36}S^3\sqrt{\frac{25+10\sqrt{5}-5\sqrt{5}-10}{600}} = \frac{5}{36}S^3\sqrt{\frac{15+5\sqrt{5}}{600}} = \frac{5}{36}S^3\sqrt{\frac{3+\sqrt{5}}{120}}$; but as there are 12 such pyramids in the body, therefore the dodecaedron $= \frac{5}{3}S^3\sqrt{\frac{3+\sqrt{5}}{120}} = \frac{5}{6}S^3\sqrt{\frac{3+\sqrt{5}}{30}}$.

Cor. *The radius of the sphere inscribed in the dodecaedron, is* $DA\sqrt{\frac{25+11\sqrt{5}}{40}}$; DA *being the side of the dodecaedron.*

For

For that radius is $= P = S\sqrt{\frac{5+2\sqrt{5}}{60}} =$ FIG. 221.
$DA\sqrt{\frac{5+2\sqrt{5}}{60}} \times \frac{9+3\sqrt{5}}{2}$ (32) $= DA\sqrt{\frac{75+33\sqrt{5}}{120}}$
$= DA\sqrt{\frac{25+11\sqrt{5}}{40}}$.

PROP. XXXIV.

The square of the diameter of a sphere, is to the square of the side of its inscribed regular icosihedron; as 10 *to* $5-\sqrt{5}$; *or as* $5+\sqrt{5}$ *to* 2. 222.

Let BDEFG be the pentagonal base of the solid angle A, made by 5 triangles of the icosiedron; let AC be perp. to it, and draw DC. Then $DC^2 = \frac{2}{5-\sqrt{5}}DE^2$ (44. IV) $= \frac{2}{5-\sqrt{5}}AD^2 = AD^2 \times \frac{2}{5-\sqrt{5}} \times \frac{5+\sqrt{5}}{5+\sqrt{5}} = \frac{10+2\sqrt{5}}{25-5}AD^2 = \frac{5+\sqrt{5}}{10}AD^2$. And $AC^2 = AD^2 - DC^2$ (Cor. 1. 21. II) $= AD^2 - \frac{5+\sqrt{5}}{10}AD^2 = \frac{5-\sqrt{5}}{10}AD^2$, and $AC = AD\sqrt{\frac{5-\sqrt{5}}{10}}$. But the diameter of the sphere $= \frac{AD^2}{AC} = \frac{AD^2}{AD\sqrt{\frac{5-\sqrt{5}}{10}}} = AD\sqrt{\frac{10}{5-\sqrt{5}}}$, and the square of the diameter $= AD^2 \times \frac{10}{5-\sqrt{5}} = SS$; and $AD^2 = \frac{5-\sqrt{5}}{10}SS = \frac{2SS}{5+\sqrt{5}}$.

Cor. 1. *The diameter of the sphere, is to the diameter of the circle comprehending five sides of the icosiedron; as* $\sqrt{5}$ *to* 2.

For if $S =$ diameter of the sphere, then $SS = AD^2 \times \frac{10}{5-\sqrt{5}}$, and $DC^2 = AD^2 \times \frac{2}{5-\sqrt{5}}$, and

FIG. 222. $4DC^2 = AD^2 \times \frac{8}{5-\sqrt{5}}$; therefore $SS : 4DC^2 :: 10 : 8 :: 5 : 4$. And $S : 2DC :: \sqrt{5} : 2$.

Cor. 2. *The square of the diameter of the sphere, is to the square of the diameter of the circle containing one triangle of the icosiedron; as* 15, *to* $10 - 2\sqrt{5}$.

For let R = radius of the circle circumscribing the triangle ADB; then $AD^2 = 3RR$ (41. IV), and $AD^2 = \frac{5-\sqrt{5}}{10}SS$ (34); therefore $\frac{5-\sqrt{5}}{10}SS = 3RR$, and $\frac{5-\sqrt{5}}{30} \times SS = RR$, and $\frac{10-2\sqrt{5}}{15}SS = 4RR$.

Cor. 3. *The same circle comprehends both the pentagon of a dodecaedron, and the triangle of an icosiedron, inscribed in the same sphere.*

Cor. 4. *The area of a triangle* ADB *of the icosiedron, is equal to* $\frac{5\sqrt{3}-\sqrt{15}}{40} \times$ *square of the diameter of the sphere.*

For the area $= \frac{DA^2}{4}\sqrt{3}$ (39. II) $= \frac{SS}{4} \times \frac{5-\sqrt{5}}{10}\sqrt{3}$ (34) $= SS \times \frac{5-\sqrt{5}}{40}\sqrt{3} = SS \times \frac{5\sqrt{3}-\sqrt{15}}{40}$.

PROP. XXXV.

222. *The cube of the diameter of a sphere, is to the solidity of the inscribed regular icosihedron; as* 6 *to* $\sqrt{\frac{5+\sqrt{5}}{2}}$.

Let P = the perpendicular from the center of the sphere, upon the triangle ADB of the icosiedron. R = radius of the circle encompassing the triangle. Then $RR = \frac{5-\sqrt{5}}{30}SS$ (Cor. 1. 34).

Then

Then $PP = \frac{1}{4}SS - RR = \frac{1}{4}SS - \frac{5-\sqrt{5}}{30}SS =$ FIG. 222.
$\frac{5+2\sqrt{5}}{60} \times SS$, and $P = S\sqrt{\frac{5+2\sqrt{5}}{60}}$. And area of the triangle $ADB = \frac{5-\sqrt{5}}{40}\sqrt{3} \times SS$ (Cor. 4. 34). Therefore the pyramid whose base is ADB, and vertex at the center of the sphere is $= \frac{1}{3}P \times ADB$ (18. VI) $= \frac{SS\sqrt{3}}{3} \times \frac{5-\sqrt{5}}{40} \times S\sqrt{\frac{5+2\sqrt{5}}{60}}$ (dividing by $\sqrt{3}$) $= \frac{S^3}{3 \times 40} \times \overline{5-\sqrt{5}} \times \sqrt{\frac{5+2\sqrt{5}}{20}}$ (squaring $5-\sqrt{5}$) $= \frac{S^3}{3\times 40} \times \sqrt{\overline{30-10\sqrt{5}} \times \frac{5+2\sqrt{5}}{20}}$ $= \frac{S^3}{3 \times 40}\sqrt{\frac{50+10\sqrt{5}}{20}} = \frac{S^3}{3 \times 40}\sqrt{\frac{5+\sqrt{5}}{2}}$. And 20 such pyramids, or the icosiedron $= \frac{S^3}{6}\sqrt{\frac{5+\sqrt{5}}{2}}$.

Cor. *The radius of the sphere inscribed in the icosihedron, is* $DA\sqrt{\frac{7+3\sqrt{5}}{24}}$, *DA being the side of the icosihedron.*

For that radius is $= P = S\sqrt{\frac{5+2\sqrt{5}}{60}} = DA\sqrt{\frac{5+2\sqrt{5}}{60} \times \frac{5+\sqrt{5}}{2}}$ (34) $= DA\sqrt{\frac{35+15\sqrt{5}}{120}}$ $= DA\sqrt{\frac{7+3\sqrt{5}}{24}}$.

SCHOLIUM.

A sphere may be inscribed or circumscribed to any regular body, or to any triangular pyramid.

BOOK VIII.

The conſtruction of geometrical problems.

DEFINITION.

FIG. A Problem is ſaid to be *conſtructed geometrically*, when it is done by the help only of a ſtraight ruler, and a pair of compaſſes.

PROB. I.

223. *To draw a ſtraight line from one point* A, *to another* B, *upon a plane.*

Set one foot of the compaſſes in the point A, and apply the edge of one end of the ruler to it; keep it cloſe there, whilſt you turn the other end of the ruler about, till the edge of it fall upon the other point B; then draw a line by the edge of the ruler, which will go from one point to the other.

PROB. II.

224. *To produce a line* AB, *that is too ſhort.*

Lay the edge of one end of the ruler againſt the foot of the compaſſes, placed at one end of the line A; and turn the other end about it, till the edge fall upon the other end of the line B. Then through B, draw a line by the edge of the ruler, from B to F.

Other-

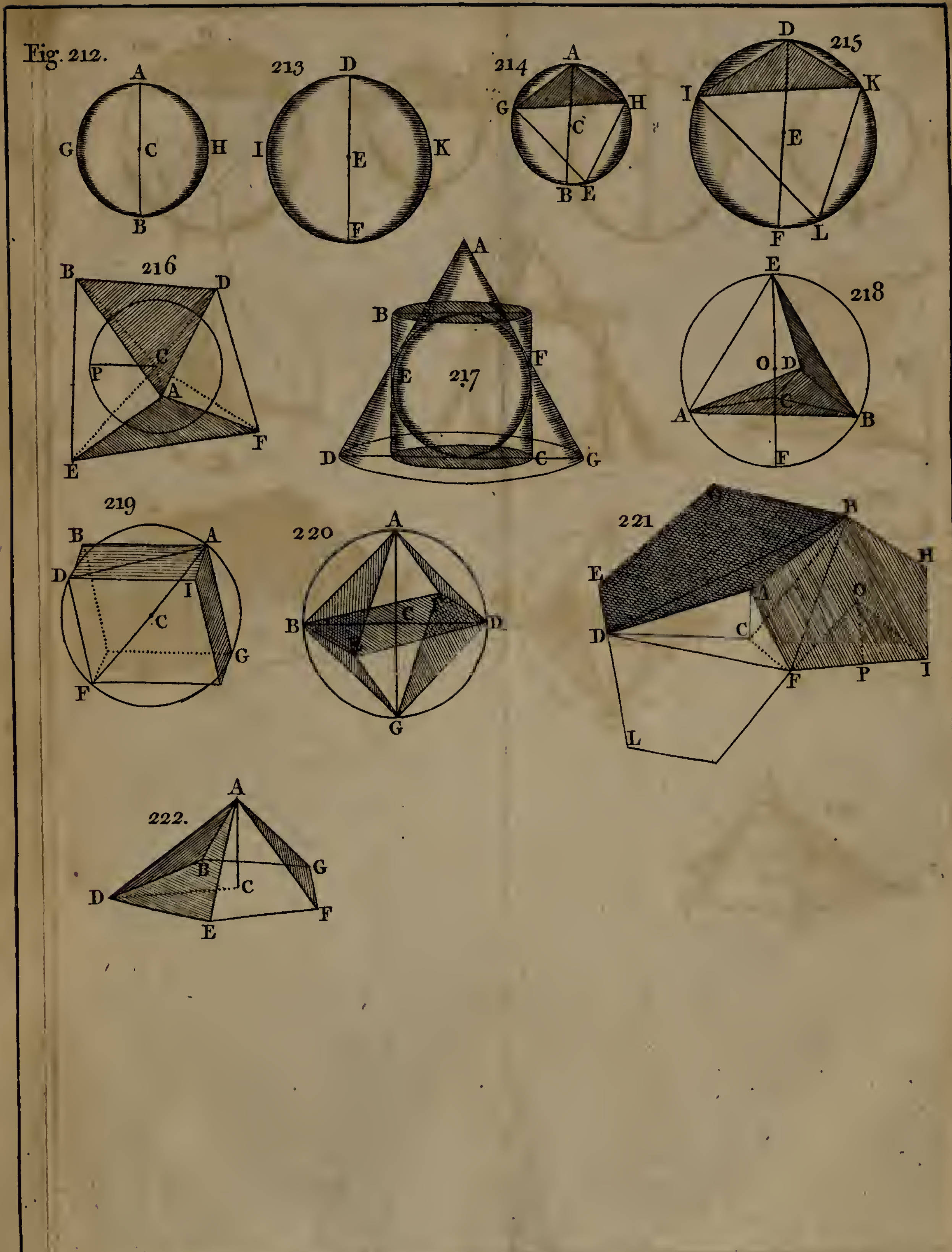
Fig. 212.
A C G H B
213
D E I K F
214
A G H C B E
215
D I K E F L
216
B D C P A E F
217
A B F E C D G
218
E O D C A B F
219
B A D I C G F
220
A B C D G
221
E H C D P I F L
222.
A B G C D E F

Pl. XI. *Pa.* 156.

Otherwise. FIG.

Place one foot of the compaſſes in the end A, and through the other end, draw the obſcure arch CBD, with the other foot of the compaſſes, opened to the diſtance AB. In that arch take BC equal to BD; then with any opening of the compaſſes, ſeting one foot in C and D, deſcribe two obſcure arches to interſect in E; then draw BEF. 224.

For if the lines AC, AD, CE, DE, CD be ſuppoſed to be drawn; the line CD will be ⊥ to AB, and to BF (Cor. 3. 3. II); and ABF, a right line (1. I).

PROB. III.

From a given point C, *to draw a line equal to a given line* AB. 225.

Draw the line CD, ſufficiently long; then take the extent AB in your compaſſes, and ſeting one foot in C, ſtrike the obſcure arch, F. Then CF = AB.

PROB. IV.

To find the ſum and difference of two given lines AB, BD. 226.

Draw any line DA ſufficiently long, then take the ſhorter line AB in your compaſſes, and ſeting one foot in B, deſcribe two arches to cut AD in A and F; then will DA = BD + BA, and DF = BD — BA.

PROB. V.

To divide a given angle ACB *into two equal parts.* 227.

From the angular point C deſcribe any arch AB, to cut CA, CB; then with any extent, ſeting one foot in A and B, deſcribe two obſcure arches, to cut

FIG. cut each other in D; then draw CD; and ∠ACD
227. = DCB.

For if AD, BD be ſuppoſed to be drawn; the ∠DCA = DCB (8. II).

PROB. VI.

228. *To divide a given right line* AB *into two equal parts.*

From the ends A, B, with the ſame extent, deſcribe two arches, to cut one another in C, and D. Draw CD to cut AB in I. Then AI = IB.

For if AC, AD, BC, BD be ſuppoſed to be drawn, ACBD will be a rhombus; and AI = IB (2. III).

PROB. VII.

229. *To make an angle* B, *equal to a given angle* A.

Upon the angular point A as a center deſcribe an arch FG. Draw any line BC, and from B with the ſame extent as before, deſcribe the arch CD. Make the arch CD = FG, and draw BD. Then ∠CBD = ∠FAG.

PROB. VIII.

230. *Through a given point* A, *to draw a line* AB *parallel to another* CD.

Take the neareſt diſtance of the point A from CD; and ſeting one foot in ſome point of the line CD, deſcribe an occult arch O. Then through A draw a line AB to touch the arch O; which will be ∥ to CD.

Otherwiſe.

231. From ſome point O in the line CD as a center, with the diſtance OA, deſcribe a ſemicircle CABD paſſing through A; then make the arch DB

DB = arch CA; and draw AB, which will be ∥ to CD (Cor. 13. IV). FIG. 231.

Or thus.

With any extent, and one foot in A, defcribe an arch to cut CD in fome point C. And with the fame extent, and one foot in fome point as D, in the line CD, defcribe an arch B to cut AB. Then with the extent CD, and one foot in A, crofs the laft arch in B; then draw AB, which is parallel to CD (1. III). 232.

Or thus.

From a point D taken at pleafure in the line DC, defcribe through A, the arch AC; and from A, with the fame extent, the arch DB. Make DB = AC. And draw AB, which will be ∥ to DC (Cor. 2. 4. I). 233.

PROB. IX.

From a given point P *in a right line* AB, *to raife a perpendicular.* 234.

Make PC equal to PB, and from C and B, with a convenient extent, defcribe two arches to cut each other at D; draw DP, which will be ⊥ to CB (8. II).

Or thus.

With any diftance PF, and one foot in P, defcribe the circle FCD, and fet FP from F to C, and from C to D; from the points C, D, with any extent, defcribe two arches to interfect at O, then draw OP, which is ⊥ to AB. 235.

For FC is the third part of a femicircle (45. IV), and CD is bifected by OP (Cor. 3. 3. II), and alfo the arch CD (Cor. 2. 2. IV), and therefore ∠FPO = OPB = a right angle.

PROB.

FIG.

PROB. X.

236. *To raiſe a perpendicular on the end A, of a line given,* AB.

Set one foot in A, and extend the other to any point C out of the line AB. From C as a center deſcribe the ſemicircle PAF, to cut AB in F. Through F and C draw FCP, to cut the ſemicircle in P. Then draw PA, which will be $\perp$ to AB (14. IV).

Otherwiſe.

237. From the center A, at any diſtance AF, deſcribe the arch FG; ſet AF from F to G. And from G with the ſame extent deſcribe an arch P. Through F and G, draw the line FGP, to cut the arch in P. Then draw PA, which is perpendicular to AB.

For if AG be drawn, $\angle$FAG $= \frac{2}{3}$ of a right angle (Cor. 2. 3. II) $=$ AGF $=$ 2GAP (1. II). Therefore GAP $= \frac{1}{3}$ a right angle; and the whole FAG $+$ GAP $= \frac{2}{3} + \frac{1}{3} =$ 1 whole right angle.

Or thus.

238. Take any length in your compaſſes, as AC; and ſet it 5 times along the line AB, to C, E, D, I, K; take 3 parts AD in your compaſſes, and with one foot in A deſcribe an arch P; then with extent AK (or 5 parts), and one foot in I, croſs the arch P; then from the point of interſection P to A draw PA, which is $\perp$ to AB (Cor. 3. 21. II).

It will be the ſame thing, if you ſet AI from A to P, and AK from D to P.

PROB.

PROB. XI.

FIG.

From a given point P, *to let fall a perpendicular upon a given line* AB. 239.

From the center P defcribe an arch to cut AB in E and F. From E and F, with a proper diftance, defcribe two obfcure arches to interfect in I, then through P and I, draw PC; which is perp. to AB (Cor. 3. 3. II).

Or thus.

From a point A in the line AB, with diftance AP, defcribe the arch PI; likewife from another point D, in AB, with diftance DP, defcribe the arch PI to cut the former in I. Draw PCI, and PC is $\perp$ to AB. 240.

For if AP, AI be drawn, then PC = CI, and AC $\perp$ PI (Cor. 3. 3. II. and 8. II).

PROB. XII.

To divide the given line AB *into any number of equal parts.* 241.

Draw any indefinite line AP, on which fet the equal parts AL, LM, MN, NP. Draw PB, and through L, M, N, draw LD, ME, NF ‖ to PB. Then AD = DE = EF = FB (12. II).

Otherwife.

From the ends A, B, of the given line, draw two lines AP, BK as long as you will, parallel to one another. Then fet any equal parts from A towards P, and likewife from B towards K. Then draw lines between the correfpondent points, NG, MH, LI, which will divide AB into the equal parts AD, DE, EF, FB (12. II). 242.

M Or

FIG.

Or thus.

243. Let AB be given to be divided; draw CP ∥ to AB. Set any equal parts, from C to L, L to M, M to N, and from N to P. Draw CA and PB to interfect in G; and draw GL, GM, GN, to cut AB in D, E, F. Then AD, DE, EF, FB are all equal (Cor. 13. II).

PROB. XIII.

244. *To divide a given line* AB, *in proportion as another line* AC *is divided in* D *and* E.

Let AB and AC be joined at A, making the angle BAC; draw CB; and through D, E, draw DF, EG ∥ to CB. Then will AF : AD :: FG : DE :: GB : EC (Cor. 2. 12. II).

PROB. XIV.

245. *To find a third proportional to two given lines,* AB, AD.

Join AB, AD at A, ſo as to make an angle BAD. Produce AD, and make AC = AD, and draw CE ∥ to BD; then AE is the third proportional. For AB : AD :: AC or AD : AE (13. II).

PROB. XV.

246. *To find a fourth proportional to three given lines,* AB, AC, AD.

Let AB, AC make any angle at A, apply the third line from A to D. Draw BC, and DE ∥ to BC; then AE is the fourth proportional. For AB : AC :: AD : AE (13. II).

PROB.

PROB. XVI.

FIG.

To find a mean proportional between two given lines AB, BD. 247.

Let AD be the ſum of the two lines AB, BD (4); biſect AD in C. With center C, and radius CA or CD, deſcribe the ſemicircle AED. From B erect BE ⊥ to AD, to cut the circle in E; then BE is the mean proportional ſought.

For AB : BE : : BE : BD (17. IV).

Or thus.

Let BA be the greater, biſect it in C, and from the center C, with radius CA or CB, deſcribe the ſemicircle BEA. Let BD be the leſſer given line. Erect DE ⊥ to BA (9), to cut the circle in E, draw BE, which is a mean between BD and BA (Cor. 17. IV). 248.

PROB. XVII.

To divide the given line AB *in extreme and mean proportion.* 249.

Draw EAF ⊥ to AB, and make AE $= \frac{1}{2}$AB, and draw EB, and make EF = EB, and AG = AF. And G is the point of diviſion.

For AF = EF — EA (Conſt.), that is, AG = EB — EA, and AG + AE = EB (Ax. 3), that is, AG + $\frac{1}{2}$AB = EB; and $AG^2 + AG \times AB + \frac{1}{4}AB^2 = EB^2$ (10. I), and $AG^2 = EB^2 - AG \times AB - EA^2$ (becauſe $\frac{1}{4}AB^2 = EA^2$) $= AB^2 - AB \times AG$ (Cor. 21. II) $= AB \times \overline{AB - AG} = AB \times BG$, therefore AB is cut in G, in extreme and mean proportion (Def. 11. Proportion).

FIG. 249. *Cor.* $AG = AB \times \frac{\sqrt{5}-1}{2}$, *and* $BG = AB \times \frac{3-\sqrt{5}}{2}$.

For EB or $EF = \sqrt{\frac{5}{4}AB^2} = \frac{AB}{2}\sqrt{5}$, and AF or $AG = EF - \frac{1}{2}AB = AB \times \frac{\sqrt{5}-1}{2}$.

Also $BG = AB - AG = AB \times \frac{2-\sqrt{5}+1}{2} = AB \times \frac{3-\sqrt{5}}{2}$.

PROB. XVIII.

250. *In any triangle* ABC, *to draw a perpendicular from any angle* A *to its oppofite fide* CB.

About either of the other fides AB, defcribe a femicircle ADB, to cut the fide CB in D. Draw AD, which will be $\perp$ to CB (14. IV).

PROB. XIX.

251. *Upon a given line* AB, *to make an equilateral triangle.*

Take AB in your compaffes, and with one foot in A and B, defcribe two arches to crofs each other at C. Draw AC, BC; and ABC is the triangle.

PROB. XX.

252. *To make a triangle of three given lines* A, B, C; *of which any two muft be greater than the third.*

Draw DE = the line A; then take B in your compaffes, and with one foot in E defcribe an occult arch F. Then take C in your compaffes, and with one foot in D, crofs the former arch at F; draw DF, EF; and DEF is the triangle required.

Cor.

Cor. *After the same manner, a triangle is made equal to a given triangle.* FIG. 252.

PROB. XXI.

To make an isosceles triangle ABD, *whose side is the given line* AB; *and angle at the base* B *or* D, *double to that at the top* A. 253.

Let AC be the greater part of the line AB divided in extreme and mean ratio (17). From the center A through B, describe the circle BD; and with extent CA, and one foot in B, cross the circle in D; and draw AD. Then ABD is the triangle sought.

For draw CD; then since AB : AC :: AC : CB (Def. 11. Proportion), that is, AB : BD :: BD : BC; therefore the triangles ABD, BDC are similar (14. II), and BD = DC = CA. Whence the ∠B or BCD = ∠A + CDA (1. II) = 2∠A (3. II).

Cor. *The angle* A *is* $\frac{2}{5}$ *of a right angle.*

PROB. XXII.

A triangle ABC *being given; to reduce it to another of a different base,* AED. 254.

Let AE be the base proposed, being in the same line AB. Draw the line CE, from the top of the given triangle, to the point E proposed. And through ∠B of the given triangle, draw BD ∥ to CE; draw the line DE. Then the triangle ADE = ACB.

For triangle DBE = triangle DBC (10. II). Add ADB, then ADB + BDE or ADE = ADB + BDC or ABC.

Cor. *Thus a triangle may be reduced to another of a different hight.*

FIG.

PROB. XXIII.

255. *To divide a triangle* ABC, *in any proportion, by a line drawn from an angle* A.

Divide the base, or opposite side BC, in D, in the proportion given (13); to D, draw the line AD; which divides the triangle ABC, in the same given ratio (11, II).

PROB. XXIV.

256. *To reduce a polygon* ABCDE *to fewer sides.*

To take away the angle B; produce the next side EA, then draw the diagonal CA, and from B, draw BG ‖ to CA, to cut EA in G; and draw CG, which takes in the triangle CAG, instead of its equal CAB (10, II). So the polygon becomes CGED.

Cor. *By thus taking away one angle after another; any polygon may, at last, be reduced to a triangle.*

PROB. XXV.

257. *Upon a given line* A, *to make a square.*

Draw BC = A, take A or BC in your compasses, and with one foot in C, describe the arch BED; and with one foot in B, the arch CEF. Set the same extent from the intersection E to F; draw CF to cut BE in G; make ED and EI = EG; and draw BI, ID, DC, and BIDC is the square required.

For if CE, BE, BF be drawn, ∠BCE = $\frac{2}{3}$ a right angle (Cor. 2. 3. II) = CBE = EBF, and ∠ECF = $\frac{1}{3}$ a right one (12. IV), therefore ECD = $\frac{1}{4}$ a right angle, and BCD = a right angle.

Or

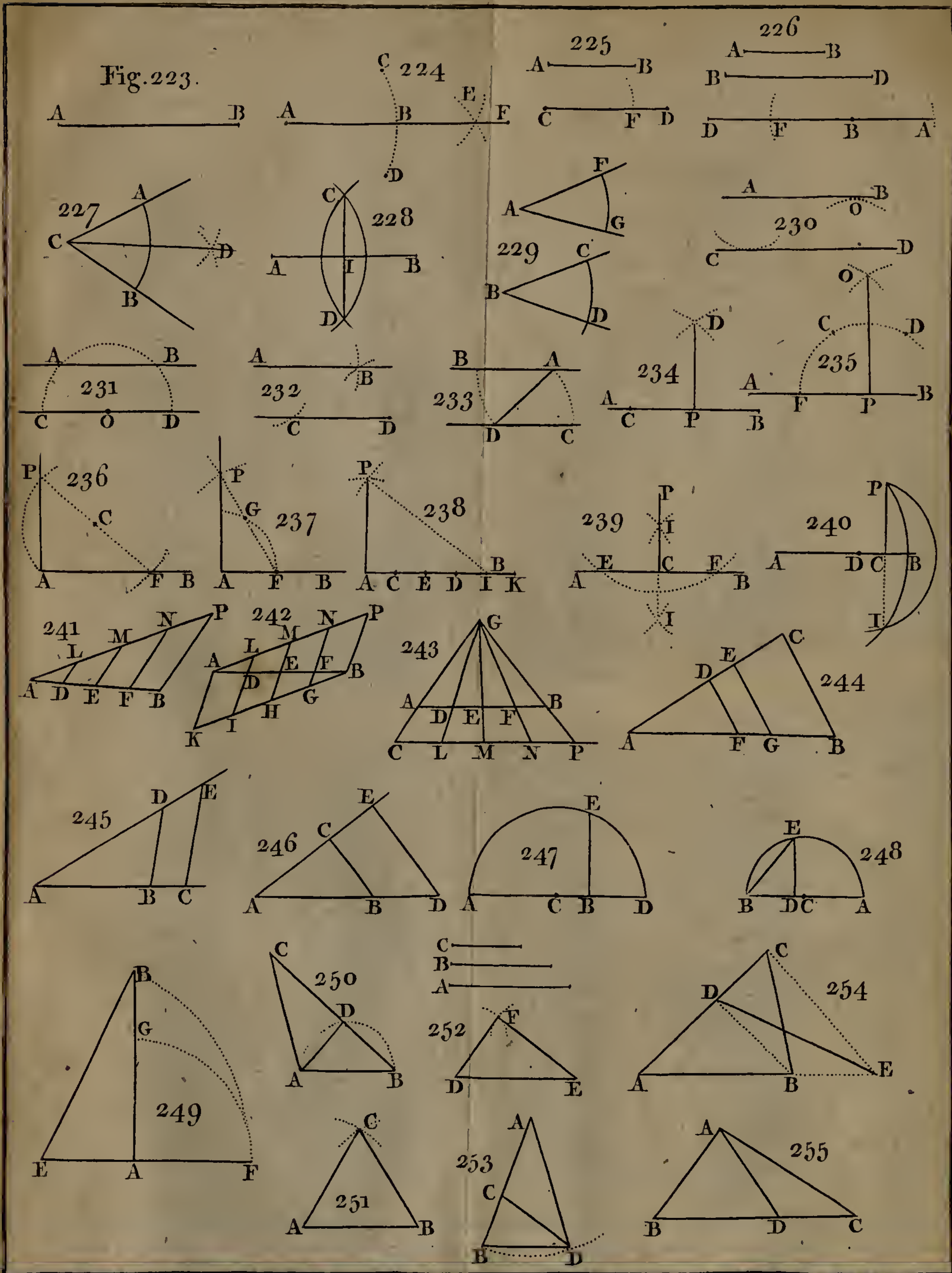
Fig. 223.
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Pl. XII. *Pa. 166.*

Or thus. FIG.

Make BC = A, draw CD ⊥ and = CB (9); with extent BC, and one foot in B, defcribe an arch I; with the fame extent and one foot in D, crofs the arch at I; draw BI, ID; then BIDC is the fquare. 257.

PROB. XXVI.

With two given lines A, B, *to make a rectangle.* 258.

Make the bafe CD = B, draw CF perp. to CD (9), and = A; with the extent B, and one foot in F, defcribe an arch E; and with extent A, and one foot in D, crofs the arch at E; draw FE, ED; and CFED is the rectangle fought.

PROB. XXVII.

To make a fquare equal to a given rectangle ABCD. 259.

Produce the fides AD, CD, and make DF = DC; bifect AF in I, and with radius IA or IF, defcribe the femicircle AEF to cut CE in E. On DE make the fquare DH, which will be equal to the rectangle AC or AD × DF (17. IV).

PROB. XXVIII.

To make a parallelogram equal to a triangle given ABC; *and having an angle, equal to a given angle* D. 260.

Through A draw AG ∥ to BC, and make the ∠BCG = D; bifect BC in E, and draw EF ∥ to CG; then the parallelogram EG = triangle ABC (7. III).

FIG.

PROB. XXIX.

261. *Upon a given right line* A, *to make a parallelogram equal to a given triangle* B; *having an angle, equal to a given one* C.

Make a parallelogram GE = triangle B (28), whofe angle G = C; produce DG, EF, DE, GF; and make FH = A; through H, draw IL ∥ to EF, to meet DE in I; draw IFK, to cut DG in K; through K draw KL ∥ to GH, meeting EF and IH in M and L. Then the parallelogram MH = B.

For parallelogram MH = GE (4. III) = B (Conftr.).

Or thus.

262. Let B be the given triangle; produce the bafe, and draw EG, parallel thereto; make the ∠DCG = C, and CI = bafe of the triangle B. Then triangle CGI = B (10. II); make CD = A, and make triangle CKD = CGI (22); bifect CK in H, draw HL, IL ∥ to CD, CH; then CL is the parallelogram fought.

For CHLD = triangle CKD (7. III) = CGI (Conftr.) = B.

PROB. XXX.

263. *Upon a given right line* FG, *to make a parallelogram equal to a given polygon* BACD, *having an angle equal to a given one* E.

Divide the polygon into triangles CAB, CBD. Make the angle GFK = E; and make the parallelogram GI = triangle CAB, and HK = CBD (29). Then parallelogram FL = CABD.

Cor.

Cor. 1. *Hence a square may be made equal to any given polygon; by making a rectangle equal to the polygon, and then a square equal to the rectangle* (27). FIG. 263.

Cor. 2. *Thus a parallelogram may be made equal to the sum or difference of two given polygons.*

PROB. XXXI.

To make a square equal to the sum of two squares. 264.

Make FBD a right angle; make BA = side of one given square; BC = side of the other square, draw AC; then the square made on AC, is equal to the sum of the squares made upon AB, and BC (21. II).

Cor. *After the same manner a square may be found equal to three or more squares. For draw* OC ⊥ *to* AC, *and equal to the side of a third square, and draw* AO. *Then* $AO^2 = AC^2 + CO^2 = AB^2 + BC^2 + CO$ (21. II); *and so on.*

PROB. XXXII.

To make a square equal to the difference of two squares. 264.

Make the right angle FBD; set the side of the lesser square from B to A; take the side of the greater in your compasses, and seting one foot in A, with the other cross the line BD, in C. Then CB is the side of the square equal to the difference of the squares (Cor. 21. II).

PROB. XXXIII.

To find the proportion of one polygon A to another B. 265.

Find two squares equal to the two polygons A, B (Cor. 1. 30); let CF be the side of the

FIG. 265. first, and draw FE ⊥ to it, and equal to the side of the second. Draw the hypothenuse CE; from F, let fall the perpendicular FD upon it: then CD : DE : : polygon A : polygon B.

For CD : DE : : CF^2 : FE^2 (Cor. 4. 20. II) : : A : B (Constr.).

PROB. XXXIV.

266. *To make a triangle equal and similar to a given triangle* ABC.

Draw any line DE, and make DE = AB; then with extent AC, and one foot in D, describe an occult arch F. And with extent BC, and one foot in E, cross the arch at F; draw DF, EF; and DEF is the triangle required (8. II).

Or thus.

Make the ∠EDF = BAC, and DE = AB, and DF = AC, and draw EF. And DEF is the triangle sought (6. II).

PROB. XXXV.

267. *To make a plane figure equal and similar to another* ABCDEF.

In any line AF, take two marks or points M, N. Also in the line *af*, take *mn* = MN. With the distances from M to B, C, D, &c, and one foot in *m*, describe as many arches; then with the distances from N to B, C, D, &c, and one foot in *n*, cross them in *b*, *c*, *d*, *e*, &c. make *ma* = MA, *nf* = NF; and draw the lines *ab*, *bc*, *cd*, *de*, *ef*, in like manner as the correspondent lines are drawn in the other figure; and it is done.

Or

Or thus. FIG.

Let the given figure ABCDE be divided into the triangles BAC, CAD, DAE. Then make triangle GFH = BAC, HFI = CAD, and IFK = DAE (34). And the polygon GK will be equal and similar to BE. 268.

PROB. XXXVI.

To make a polygon similar to another ABCDE, *and in the given ratio of* AF *to* AB. 269.

Find AG a mean proportional between AF and AB. Draw the diagonals AC, AD. Then from G, draw GH, HI, IK parallel to BC, CD, DE. And AGHIK is the polygon.

For the correspondent triangles in both being similar, the polygons are similar (Cor. 2. 19. III). Also AF : AG :: AG : AB (Constr.), and AF : AB :: AG^2 : AB^2 (23. Proportion) :: polygon GI : polygon BD (20. III).

Otherwise thus.

Make PQ = AG; also make the angles QPR, RPS, SPT, respectively equal to BAC, CAD, DAE. And make the angles Q, R, S, T successively = B, C, D, E. And the polygon PQRST is that sought. 269. 270.

For each of the triangles in one figure is similar to each in the other (Def. 10. II); and therefore the polygons are similar (Cor. 1. 19. III).

Cor. *And thus a polygon is made upon a given line* AG *or* PQ, *similar to another polygon* ABCDE.

PROB.

FIG.

PROB. XXXVII.

270. *To make a polygon* TQ *equal to a given one* F, *and*
271. *similar to another* ACDEB.
272.

Upon BA make the rectangle BM = BACDE (30); and upon BH make the rectangle BI = F (30). Take PQ a mean proportional between BA and BR (16); and upon PQ, make the polygon PQRST similar to BACDE (Cor. 36); and TQ is the polygon sought = F.

For polygon BD : polygon PS : : $BA^2 : PQ^2$ (20. III) : : BA : BR (23. Proportion) : : BM or polygon BD : BI or polygon F (8. III). Therefore polygon PS = F (Ax. 7. Proportion).

Cor. *If the polygon* TQ *was to be to* F, *in the given ratio of* R *to* S; *it is done the same way*; *only the parallelogram* BI *must be made* $= \frac{R}{S} \times F$.

PROB. XXXVIII.

To find the center of a circle ADF.

273. Draw any line AD, and bisect it in B; through B draw GBF ⊥ to AD. Bisect GF in C, for the center (Cor. 1. 2. IV).

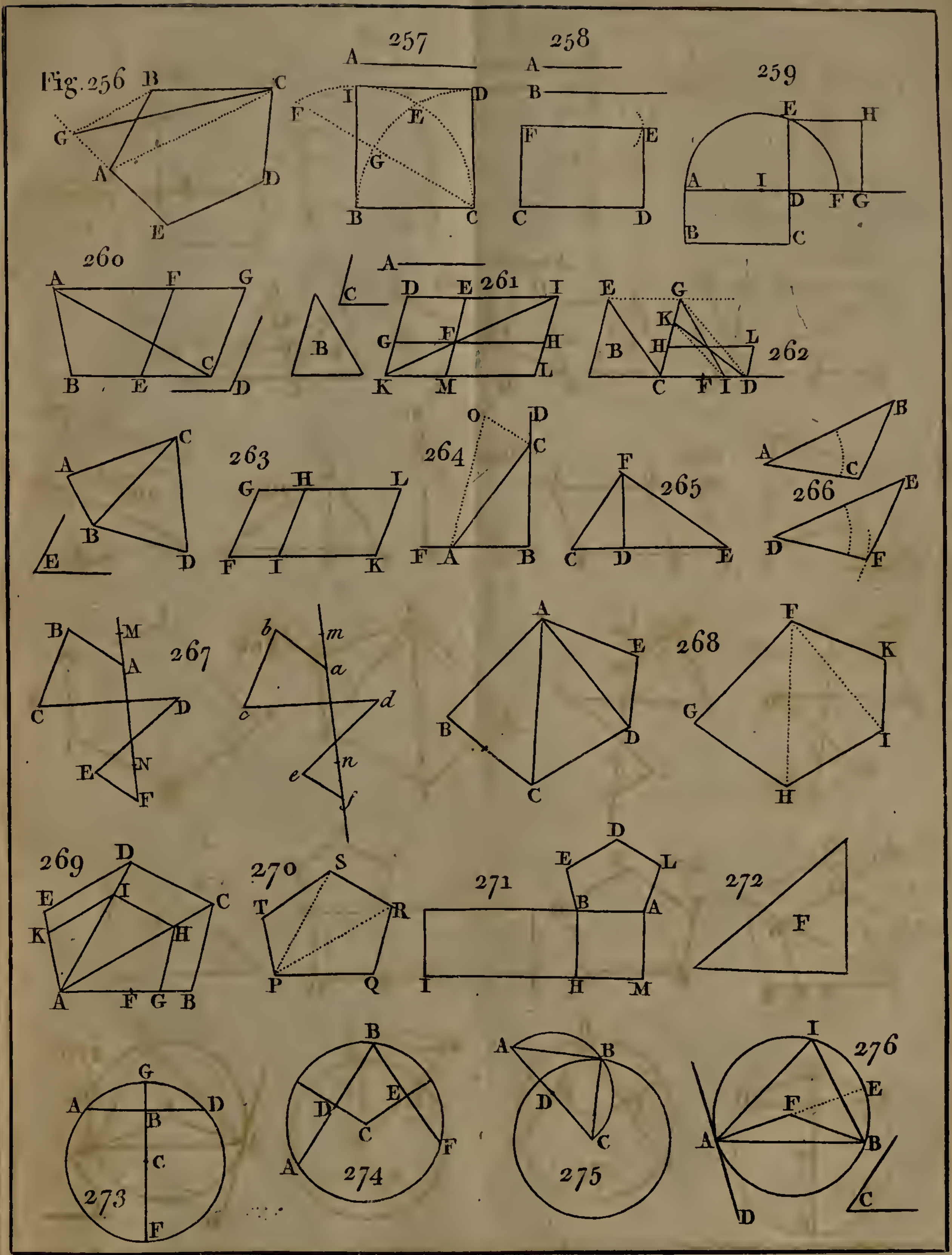
Cor. *By the same rule, the arch* AGD *is bisected* (Cor. 1. 2. IV).

PROB. XXXIX.

274. *Through three given points* A, B, F, *to describe a circle.*

Draw AB, BF, and bisect them in D, E. Thro' D and E, draw the perpendiculars DC, EC, to meet in C. C is the center (Cor. 1. 2. IV).

Cor.



Pl. XIII. *Pa.* 274.

Cor. *If an arch of a circle be given; the center may be found, by taking three points in that arch. And then the circle may be compleated.* FIG. 274.

PROB. XL.

To draw a tangent to a circle from a given point A. 275.

From the point A to the center C, draw the line AC, bisect AC in D. With the radius DA or DC, describe a semicircle to cut the circle in B. Draw AB, which will touch the circle in B (10 and 14. IV).

PROB. XLI.

Upon a right line AB, *to describe the segment of a circle, which shall contain an angle* AIB, *equal to a given angle* C. 276.

Make the angle BAD = C. Through A draw AE ⊥ to AD. At the other end B, make the ∠ ABF = BAF, to cut AE in F. From the center F, with radius FA or FB, describe the segment of a circle AIEB. Then ∠ AIB = C.

For AF = FB (Cor. 1. 3. II); and ∠ AIB = BAD (16. IV) = C.

Or thus.

Cut out a piece of wood, *&c.* with two straight sides, making an angle equal to C. And placing it between the fixt points A, B; move the angular point about, while the sides move close by the points A, B; then the angular point will describe the arch AIEB.

Cor. *In the same manner a segment is cut off from a circle, to contain a given angle; by drawing the tangent* AD *at* A, *and making the angle* BAD = C. *Then* AIEB *is the segment.*

PROB.

FIG.

PROB. XLII.

277. *In a circle* AEC, *to inſcribe a triangle ſimilar to a triangle given*, DFG.

Draw LK to touch the circle at A; make ∠KAC = F, and ∠LAB = G. Draw BC, and the triangle BAC is ſimilar to FDG (16. IV).

PROB. XLIII.

278. *In a given triangle* ABC, *to inſcribe a circle.*

Biſect two angles B, C, with the lines BD, CD, meeting in D. Let fall DF ⊥ to BC. With radius DF, and center D, deſcribe the circle FEG, which will touch all the ſides of the triangle ABC (Cor. 1. 35. II).

PROB. XLIV.

279. *About a given circle* ABC, *to deſcribe a triangle ſimilar to a triangle given*, DEF.

Produce the ſide EF both ways, to G and H. At the center I, make ∠AIB = GED, and BIC = DFH. Then to the points B, A, C, draw three tangents to the circle, to interſect in the points L, M, N. Then the triangle LMN, is ſimilar to EFD.

For ſince the ∠s at A, B, C are right angles; ∠L + AIB = 2 right angles (Cor. 16. III) = GED + DEF, and taking away the equal angles AIB, and GED; then ∠L = DEF. For the ſame reaſon M = DFE, conſequently N = D.

PROB.

PROB. XLV.

FIG. 280.

About a triangle ABC, *to deſcribe a circle.*

Biſect any two ſides, AB, BC, in D and E. Raiſe the perpendiculars DF, EF, to interſect in F. From F as a center deſcribe a circle through B, which will paſs through A, C (Cor. 32. II).

Cor. *In an acute-angled triangle, the center is within the triangle; in an obtuſe one, without* (Cor. 1. 14. IV).

PROB. XLVI.

In a given circle FCD, *to inſcribe an equilateral triangle.* 281.

Draw the diameter FB. With the radius BA and center B, deſcribe two arches C, D, to cut the circle in C and D. Draw the lines CD, DF, FC. And CFD is an equilateral triangle.

For arch CB or BD $= \frac{1}{6}$ the circumference (45. IV); therefore CBD $= \frac{1}{3} =$ CF $=$ FD.

PROB. XLVII.

In a given circle ABCD, *to inſcribe a ſquare, or regular octagon.* 282.

Draw the diameters AC, BD at right angles to one another, cuting the circle in A, B, C, D. Draw the lines AB, BC, CD, DA; and ABCD is a ſquare (Cor. 2. 6. IV).

If the diameters FG, HI be drawn, biſecting the arches AFB, AHD, DGC, CIB. Then AF, or FB, &c. will be the ſide of the octagon.

Cor. *If* AF, FB, &c. *be biſected, a polygon of* 16 *ſides, will be inſcribed; and ſo on.*

4 PROB.

FIG.

PROB. XLVIII.

283. *In a given circle* ADBG, *to inscribe a regular pentagon, or decagon.*

Draw the diameter AB; from the center C draw CD $\perp$ to AB; bisect CB in E; and make EF = ED, and draw DF, which will be the side of the pentagon; therefore if DH, HG, &c. be made = DF, DHGIK will be the pentagon required. Also FC is the side of the decagon; therefore if DL, LK, &c. be made = CF; a regular decagon will be inscribed.

For $DF^2 = DE^2 + EF^2 - 2FE \times EC$ (23. II) $= 2DE^2 - 2DE \times EC = 2DE^2 - DE \times DC$. But $DE^2 = DC^2 + CE^2$ (21. II) $= \frac{5}{4}DC^2$, and $DE = \frac{1}{2}DC\sqrt{5}$. Therefore $DF^2 = \frac{5}{2}DC^2 - \frac{DC^2}{2}\sqrt{5} = DC^2 \times \frac{5-\sqrt{5}}{2}$. Therefore DF is the side of a pentagon (44. IV). And FC is the side of a decagon (48. IV).

PROB. XLIX.

284. *In a given circle* ACE, *to inscribe a regular hexagon.*

Make AB, BC, CD, DE, EF and FA, all equal to the radius AG: and drawing the lines, the figure ABCDEF is a hexagon (45. IV).

Cor. *If the arch* AB *be bisected, you will have the side of a regular dodecagon.*

PROB.

PROB. L.

FIG. 285.

About a given circle ABC, to deſcribe a regular polygon.

Either inſcribe a polygon of the ſame ſort, or divide the circle into ſo many equal parts AB, BC, &c. as the polygon has ſides. To the points of diviſion, draw the radii GA, GB, GC, &c. To theſe lines at A, B, C, &c. draw tangents to the circle, KD, DE, EF, &c. to interſect in D, E, F, &c. then DEFHIK is the polygon required.

For if GD, GK be ſuppoſed to be drawn, AK = AD, and ∠K = ∠D (7. II). Alſo DA = DB (Cor. 4. 21. IV), whence KD = DE = EF, &c.

PROB. LI.

To inſcribe a circle in any regular polygon; or deſcribe a circle about it. 285.

Biſect any two adjoining angles D, K, with the lines DG, KG, and they will meet in the center G. Or biſect any two adjoining ſides, DK, DE, with the perpendiculars AG, BG, which will meet in the center G. Take GA the neareſt diſtance to any ſide, and from G deſcribe the circle ABC, which will touch all the ſides of the polygon DH.

Likewiſe biſect any two angles A, B, with the lines AG, BG, which will meet in the center G. Or biſect any two ſides CD, DE, with two perpendiculars meeting in G, the center. Then from A with diſtance GA deſcribe a circle ABCE, which will paſs through all the angles of the figure. 284.

Cor. *A circle may be inſcribed, or circumſcribed, to any regular polygon.*

FIG.

PROB. LII.

286. 287. *To describe a polygon in one circle* ABDE, *which shall be similar to a polygon* FGI, *described in another,* GIK; *regular or irregular.*

Draw lines from the center P, to all the angles of the polygon, as PF, PK, PI, *&c.* Then at the center O, of the other circle, make the angles AOE, EOD, DOC, COB, BOA, respectively equal to FPK, KPI, IPH, HPG, GPF. Draw lines between the points A, E, D, *&c.* Then ABCDE is similar to FGHIK (Cor. 1. 19. III).

Cor. *After the same manner, a polygon may be described about one circle, similar to a polygon described about another circle.*

PROB. LIII.

288. *From a given point* A *on high; to let fall a perpendicular to a plane* BC.

In the plane BC draw any line DE. From A draw AF ⊥ to DE. Through F, draw FH ⊥ also to DE. Then let fall AI perp. to FH. Then AI is ⊥ to the plane BC.

For DE is ⊥ to the plane AFI (4. V). And if KL be ∥ to DE, then KL is ⊥ to the plane AFI (6. V). Therefore AI is ⊥ to the plane HIL or BC (4. V).

Otherwise thus.

289. Describe a circle BFD, from the point A, upon the plane, with a pair of compasses or a string. Then find the center C of that circle (37, 38. VIII); and AC is ⊥ to the plane. In practice you need only extend from A, to three points of the plane, B, D, F.

PROB.

PROB. LIV.

FIG.

From a given point A, *in a plane* BC, *to raiſe a perpendicular.* 290.

From ſome point D, above the plane, draw DE ⊥ to the plane (52). Draw AE, and draw AF ∥ to ED. Then AF is perp. to the plane BC (6. V).

Both this and the laſt Prob. may eaſily be done with two ſquares: ſetting them croſs one another, and both of them cloſe to the point A.

PROB. LV.

To draw one plane parallel to another DE, *at a given diſtance.* 291.

Take three points A, B, C in the plane DE, but not in a right line. At theſe points erect three perpendiculars AI, BK, CL, to the plane DE (53); and of equal lengths, the ſame as the given diſtance. Through I, K, L, draw the plane FG, and it will be parallel to DE.

PROB. LVI.

To draw a plane perpendicular to a right line AB, *at* B. 292.

Draw two lines CD, EF perp. to AB at B. Through C, E, D, draw the plane CEDF, which is ⊥ to AB (4. V).

PROB. LVII.

FIG. 293. *Through any two lines* AB, CD, *inclined to one another, which do not interſect; to draw two planes perpendicular to one another.*

Through any point E of the line AB, draw EF ∥ to CD. Through the lines AEF, let the plane AEBF be drawn. From any point C, in the line CD, let fall the perp. CI, upon the plane AFB. Draw IH ∥ to FE, to interſect AB in H. At H let fall HG ⊥ to CD. Then the plane CIHG will be perp. to the plane AFH.

For CD is ∥ to IH (8. V). And ſince CI is perp. to IH, it is alſo ⊥ to CG (3. I). Therefore CI, HG are parallels (Cor. 3. 4. I); and HG ⊥ to the plane AFB (6. V). Therefore the plane DCIH is perp. to the plane AFB (7. Def. V).

Cor. 1. *The right line* GH *is perpendicular to both lines* AB, CD.

For it is ⊥ to CD (Conſtr.), and it is ⊥ to the plane AHI, and therefore to AHB.

Cor. 2. GH *is the neareſt diſtance between the two lines* AB *and* CD.

For the point H is nearer G, than any other point in the line AB (Cor. 4. 21. II). And G is nearer H than any point in CD.

Cor. 3. *Hence no two lines can poſſibly be drawn; but another line may be drawn, which is perp. to them both.*

Cor. 4. *And no two lines can be drawn, but two planes may be drawn through them, perpendicular to one another.*

Cor.

Cor. 5. *The given line* CD, *is parallel to the plane* AFB, *passing through the other line* AB. FIG. 293.

For it is parallel to HI.

PROB. LVIII.

Through any two inclined lines, which cut not one another, AB, CD; *to draw two parallel planes through them.* 293.

Draw the plane HICD and BIFA perp. to one another, and paſſing through the two given lines AB, CD (56). Then through CG at the diſtance GH, draw a plane ⊥ to GH (54), and it will be parallel to the plane ABF (Def. 10. V).

Cor. 1. *The line* GH, *(which is perpendicular to both the given lines,* AB, CD*), is the diſtance of the two parallel planes.*

Cor. 2. *No two lines can be drawn, but there may be two planes drawn through them, parallel to one another.*

PROB. LIX.

To make a ſolid angle BAD, *of three given plane angles, whoſe ſum is leſs than four right angles, and any two greater than the third.* 294.

There is no more to do than to join all their ſides AB, AC, AD, together; ſo that the vertices or angular points may all meet together in A; then A is the ſolid angle required (Cor. 19. V).

FIG.

PROB. LX.

295. *To make a ſolid angle, equal to any ſolid angle given,* A.

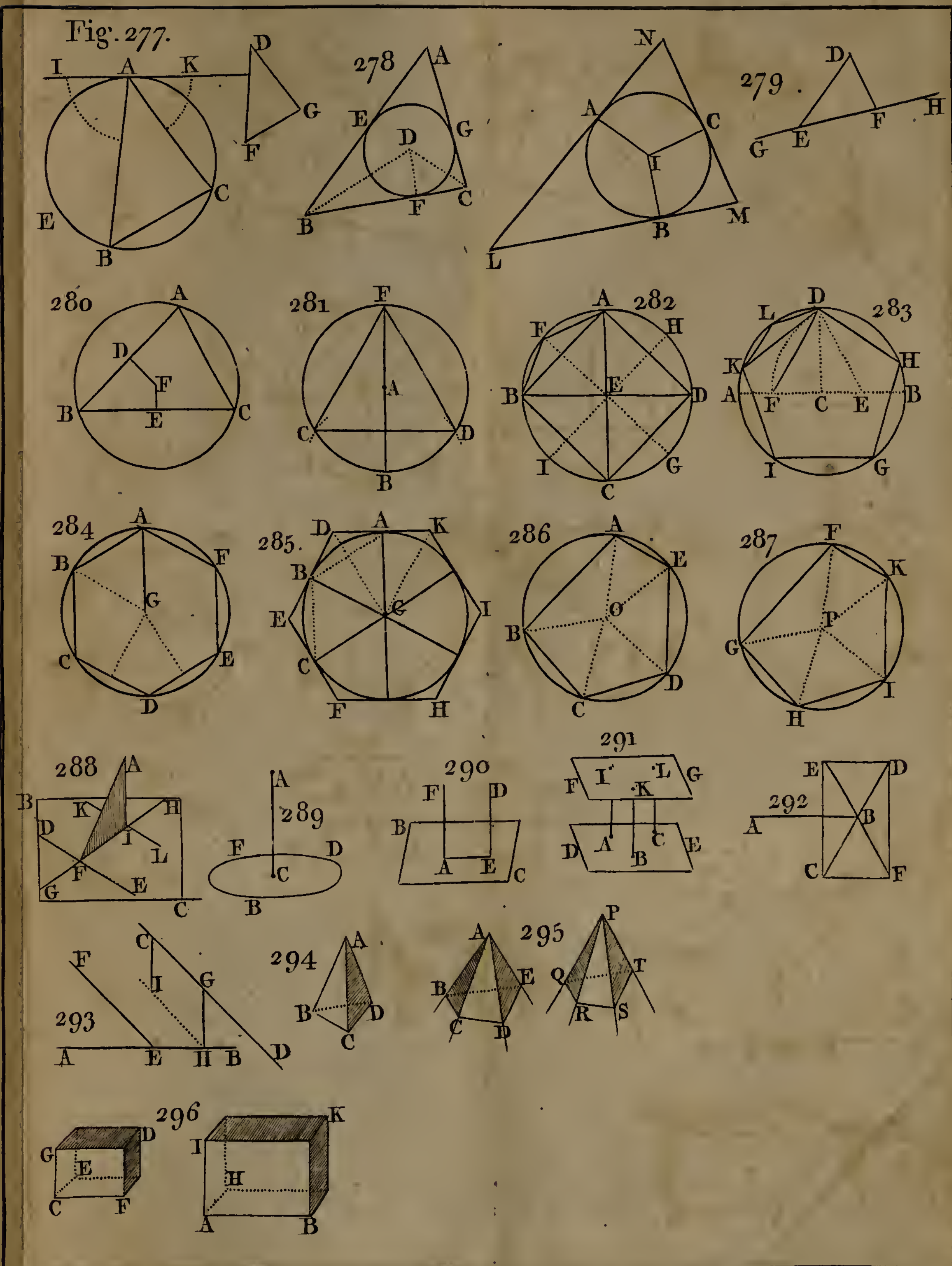
Cut off the given ſolid angle A, by a plane BCDE; and from the given planes, make the angles QPR, RPS, SPT, and TPQ reſpectively equal to BAC, CAD, DAE, and EAB; alſo make PQ, PR, PS, PT reſpectively equal to AB, AC, AD, AE. Then the plane triangles in one, will be equal to the triangles in the other. Then place the ſides PR, PS, *&c.* together as in the other ſolid angle A, ſo that all their angular points may meet in P; and likewiſe ſo that the angles Q, R, S, T, may be reſpectively equal to B, C, D, E. And then the ſolid angle P will be equal to the ſolid angle A.

For all the 3 angles at Q, being equal to thoſe at B; and all the three angles at R, equal to thoſe at C, *&c.* The ſolid angles at B, C, D, E, will be equal to thoſe at Q, R, S, T (Cor. 19. V). And conſequently $\angle$P muſt be equal to A.

PROB. LXI.

296. *Upon a given line* AB, *to deſcribe a parallelopipedon, ſimilar to a given parallelopipedon* CD.

Make the ſolid angle A equal to the ſolid angle C (59); alſo make as CF : CE : : AB : AH; and CF : CG : : AB : AI. Then finiſh the parallelopipedon AK, by drawing the planes KI, KH, and KB, parallel



Pl. XIV. *Pa. last*

parallel to the opposite ones BH, BI, and IH. Then IB is similar to GF. FIG. 297.

For their solid angles, are equal, and the sides proportional, and therefore they are similar (22. VI).

FINIS.

ERRATA.

Page	*Line*	*Read*
107	16	DFKHCILG =
121	2	Fig. 191.
	25	DCH, be 3 ——Fig. 192.
125	18	Fig. 195.
126	2, 8	Fig. 195, 196, 197, 198.
	19, 29	Fig. 195, 196, 197, 198.
127	2	Fig. 198.
128	2	Fig. 198.
	28, 9	Fig. 199, 200.
129	1, 2	Fig. 199, 200.
137	5	Fig. 209.

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